Computable and operationally meaningful multipartite entanglement measures

J.L. Beckey [1, 2, 3, 4], N. Gigena [5], P. Coles [1, 4] M. Cerezo [1, 4, 6]

[1] Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA.
[2] JILA, NIST and University of Colorado, Boulder, Colorado 80309, USA.
[3] Department of Physics, University of Colorado, Boulder, Colorado 80309, USA.
[4] Quantum Science Center, Oak Ridge, TN 37931, USA.
[5] Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland. [6] Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA.

Table of Contents

Table of Contents

Entanglement

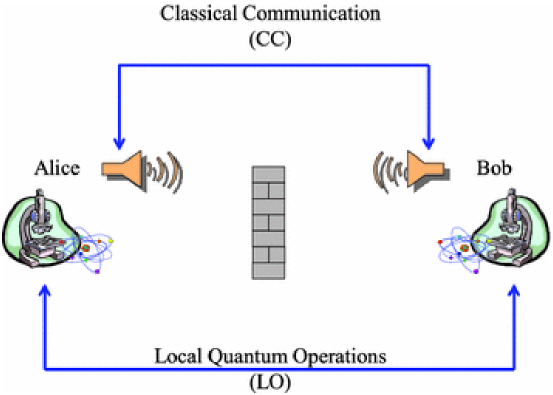


Image from M. B. Plenio, S. Virmani, Quant. Inf. Comp. 7, 1 (2007).

- **W**hen dynamics is restricted to LOCC operations the entanglement in a quantum state $|\psi\rangle$ becomes a resource.

Entanglement measures

Quantifying entanglement

- Operational approach: Link relative entanglement content to conversion rate between states $\rho \to \sigma$ via LOCC protocols.
  - Entanglement Distillation.
  - Entanglement Cost.

Quantifying entanglement

- Operational approach: Link relative entanglement content to conversion rate between states $\rho \to \sigma$ via LOCC protocols.
  - Entanglement Distillation.
  - Entanglement Cost.

- Axiomatic approach: postulate a list of properties that should be satisfied by a function $f : \mathcal{H} \to \mathbb{R}$ in order to serve as an entanglement quantifier.

Entanglement monotones

Entanglement monotones

Functions $f : \mathcal{H} \to \mathbb{R}$ that vanish for separable states and are non increasing under LOCC maps.

Entanglement monotones

Entanglement monotones

Functions $f : \mathcal{H} \to \mathbb{R}$ that vanish for separable states and are non increasing under LOCC maps.

- $f(\rho) = 0$ iff $\rho = \sum_i p_i \bigotimes_{k=1}^n \rho_k$.
- For any LOCC map described by Kraus operators $\{\mathcal{M}^j\}$ we demand

$$f(\rho) \geq \sum_j p_j f\left(\frac{1}{p_j}\mathcal{M}^j \rho \mathcal{M}^{j\dagger}\right),$$

with $p_j = \mathrm{Tr}\,(\mathcal{M}^{j\dagger}\mathcal{M}^j \rho)$.

Local operations and local eigenvalues

- Let Φ be a separable CPTP map described by Kraus operators $\{\mathcal{M}^{\boldsymbol{j}} = M_1^{j_1} \otimes \ldots \otimes M_k^{j_k} \otimes \ldots \otimes M_n^{j_n}\}$.

$$\Phi(\rho) = \sum_{\boldsymbol{j}} \mathcal{M}^{\boldsymbol{j}} \rho \mathcal{M}^{\boldsymbol{j}\dagger} = \frac{1}{p_j} \sum_{\boldsymbol{j}} \rho_{\boldsymbol{j}}$$

$$\rho \longrightarrow \rho^{\boldsymbol{j}} = \frac{\mathcal{M}^{\boldsymbol{j}} \rho \mathcal{M}^{\boldsymbol{j}\dagger}}{p_{\boldsymbol{j}}}; \quad p_{\boldsymbol{j}} = \mathrm{Tr}\left(\mathcal{M}_k^{\boldsymbol{j}\dagger} \mathcal{M}_k^{\boldsymbol{j}} \rho\right)$$

Local operations and local eigenvalues

- Let Φ be a separable CPTP map described by Kraus operators $\{\mathcal{M}^{\boldsymbol{j}} = M_1^{j_1} \otimes \ldots \otimes M_k^{j_k} \otimes \ldots \otimes M_n^{j_n}\}$.

$$\Phi(\rho) = \sum_{\boldsymbol{j}} \mathcal{M}^{\boldsymbol{j}} \rho \mathcal{M}^{\boldsymbol{j}\dagger} = \frac{1}{p_j} \sum_{\boldsymbol{j}} \rho_{\boldsymbol{j}}$$

$$\rho \longrightarrow \rho^{\boldsymbol{j}} = \frac{\mathcal{M}^{\boldsymbol{j}} \rho \mathcal{M}^{\boldsymbol{j}\dagger}}{p_{\boldsymbol{j}}}; \quad p_{\boldsymbol{j}} = \mathrm{Tr}\left(\mathcal{M}_k^{\boldsymbol{j}\dagger} \mathcal{M}_k^{\boldsymbol{j}} \rho\right)$$

- If ρ_k and $\rho_k^{\boldsymbol{j}}$ are the reduced states at site k before and after Φ:

$$\boldsymbol{\lambda}(\rho_k) \prec \sum_j p_j \boldsymbol{\lambda}(\rho_k^j)$$

Here $\boldsymbol{u} \prec \boldsymbol{v}$ means $\boldsymbol{u}$ *is majorized by* $\boldsymbol{v}$.

Building an entanglement monotone

Define a map from the state space to real numbers in the following way.

- For pure states $|\psi\rangle$:

$$E(|\psi\rangle) = S(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_N)$$

  where $\{\boldsymbol{\lambda}_k\}$ is the set of vectors of local eigenvalues and S a Schur-concave function of its arguments.

- For mixed states ρ:

$$E(\rho) = Min_{\{|\psi_i\rangle, p_i\}} \sum_i p_i E(|\psi_i\rangle)$$

  with $\sum_i p_i = 1$ and $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

E is an entanglement monotone

Case: linear entropy of entanglement

- Let $|\psi\rangle$ be a N-partite state and $\boldsymbol{L} = \{1, ..., N\}$ be the set of party labels. The linear entropy of entanglement $\mathcal{L}$ is defined for pure states as

$$\mathcal{L}(|\psi\rangle) = 1 - \frac{1}{N} \sum_{j \in \boldsymbol{s}} \mathrm{Tr} \rho_j^2$$

- $\mathcal{L}(|\psi\rangle) = \mathcal{L}(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_N)$.
- $\mathcal{L}$ is a Schur-concave function of the eigenvalue vectors $\{\boldsymbol{\lambda}_k\}$.

Table of Contents

A family of entanglement monotones

- Consider again a state $|\psi\rangle$ of a N-partite system and the set $\boldsymbol{L}$ of party labels.

- Let also $\boldsymbol{s} \subset \boldsymbol{L}$ and $\mathcal{P}(\boldsymbol{s}) = \{\alpha|\alpha \subset \boldsymbol{s}\}$.

A family of entanglement monotones

- Consider again a state $|\psi\rangle$ of a N-partite system and the set $\boldsymbol{L}$ of party labels.

- Let also $\boldsymbol{s} \subset \boldsymbol{L}$ and $\mathcal{P}(\boldsymbol{s}) = \{\alpha | \alpha \subset \boldsymbol{s}\}$.

Concentratable entanglements

$$\mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) = 1 - \frac{1}{2^{c(\boldsymbol{s})}} \sum_{\alpha \in \mathcal{P}(\boldsymbol{s})} \mathrm{Tr}\, \rho_\alpha^2$$

Here $c(\boldsymbol{s})$ is the cardinality of set $\boldsymbol{s}$.

Properties

In the following, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are subsets of the set of party labels $\boldsymbol{S} = \{1, ..., N\}$:

1. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ is a multipartite entanglement monotone.

Properties

In the following, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are subsets of the set of party labels $\boldsymbol{S} = \{1, ..., N\}$:

1. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ is a multipartite entanglement monotone.
2. AME states are maximally entangled states acording to $\mathcal{C}(\boldsymbol{L})$.

Properties

In the following, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are subsets of the set of party labels $\boldsymbol{S} = \{1, ..., N\}$:

1. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ is a multipartite entanglement monotone.
2. AME states are maximally entangled states acording to $\mathcal{C}(\boldsymbol{L})$.
3. $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leq \varepsilon \to |\mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) - \mathcal{C}_{|\phi\rangle}(\boldsymbol{s})| \leq 2\varepsilon$.

Properties

In the following, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are subsets of the set of party labels $\boldsymbol{S} = \{1, ..., N\}$:

1. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ is a multipartite entanglement monotone.
2. AME states are maximally entangled states acording to $\mathcal{C}(\boldsymbol{L})$.
3. $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leq \varepsilon \rightarrow |\mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) - \mathcal{C}_{|\phi\rangle}(\boldsymbol{s})| \leq 2\varepsilon.$
4. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s} \cup \boldsymbol{s}') \leq \mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) + \mathcal{C}_{|\psi\rangle}(\boldsymbol{s}')$ for $\boldsymbol{s} \cap \boldsymbol{s}' = \emptyset$.

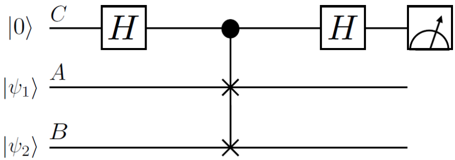
Properties

In the following, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are subsets of the set of party labels $\boldsymbol{S} = \{1, ..., N\}$:

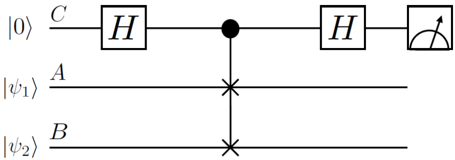
1. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ is a multipartite entanglement monotone.
2. AME states are maximally entangled states acording to $\mathcal{C}(\boldsymbol{L})$.
3. $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leq \varepsilon \rightarrow |\mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) - \mathcal{C}_{|\phi\rangle}(\boldsymbol{s})| \leq 2\varepsilon$.
4. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s} \cup \boldsymbol{s}') \leq \mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) + \mathcal{C}_{|\psi\rangle}(\boldsymbol{s}')$ for $\boldsymbol{s} \cap \boldsymbol{s}' = \emptyset$.
5. $\mathcal{C}_{|\psi\rangle}(\boldsymbol{s}') \leq \mathcal{C}_{|\psi\rangle}(\boldsymbol{s})$ if $\boldsymbol{s}' \subseteq \boldsymbol{s}$.

Table of Contents

The swap test



The swap test

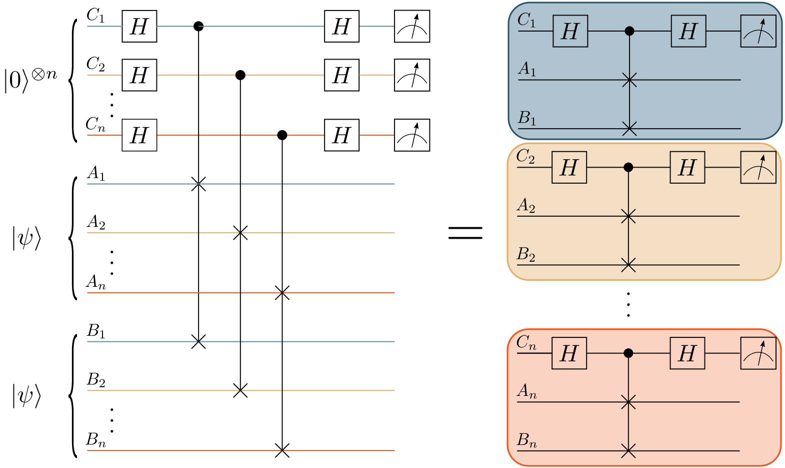


$$U_S|0\rangle|\psi_1\rangle|\psi_2\rangle = \frac{1}{2}\sum_{z=0}^{1}|z\rangle(I+(-1)^z S)|\psi_1\rangle|\psi_2\rangle$$

- Probability of finding the control qubit in the state $|z\rangle, z = 0, 1$:

$$p(z) = \frac{1}{2}(1 + (-1)^z|\langle\psi_1\,|\,\psi_2\rangle|^2)$$

Parallelized SWAP test



Parallelized SWAP test

Circuit unitary:

$$U_S^{\otimes n}|\mathbf{0}\rangle|\Psi\rangle = \frac{1}{2^n}\sum_{z=0}^{1}\prod_k |z_k\rangle(I_k + (-1)^{z_k}S_k)|\Psi\rangle$$

The probability of finding the control system in the state $|\boldsymbol{z}\rangle$, with $\boldsymbol{z} = (z_0, z_1, ...z_n)$, $z_j = 0, 1$, is

$$p(\boldsymbol{z}) = \frac{1}{2^n}\langle\Psi|\prod_k(I_k + (-1)^{z_k}S_k)|\Psi\rangle$$

Parallelized SWAP test

Circuit unitary:

$$U_S^{\otimes n}|\mathbf{0}\rangle|\Psi\rangle = \frac{1}{2^n}\sum_{z=0}^{1}\prod_k |z_k\rangle(I_k + (-1)^{z_k}S_k)|\Psi\rangle$$

The probability of finding the control system in the state $|\boldsymbol{z}\rangle$, with $\boldsymbol{z} = (z_0, z_1, ...z_n)$, $z_j = 0, 1$, is

$$p(\boldsymbol{z}) = \frac{1}{2^n}\langle\Psi|\prod_k(I_k + (-1)^{z_k}S_k)|\Psi\rangle$$

Let

- $\boldsymbol{L} = \{1, ..., N\}$.
- $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{L})$.
- $c_{w\alpha} = \#$ of 1's in $\boldsymbol{\alpha}$.

Let

- $\boldsymbol{L} = \{1, ..., N\}$.
- $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{L})$.
- $c_{w\alpha} = \#$ of 1's in $\boldsymbol{\alpha}$.

Expanding the product in

$$p(\boldsymbol{z}) = \frac{1}{2^n} \langle \Psi | \prod_k (I_k + (-1)^{z_k} S_k) | \Psi \rangle$$

Let

- $\boldsymbol{L} = \{1, ..., N\}$.
- $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{L})$.
- $c_{w\alpha} = \#$ of 1's in $\boldsymbol{\alpha}$.

Expanding the product in

$$p(\boldsymbol{z}) = \frac{1}{2^n} \langle \Psi | \prod_k (I_k + (-1)^{z_k} S_k) | \Psi \rangle$$

$$p(\boldsymbol{z}) = \frac{1}{2^n} \sum_{\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{L})} (-1)^{c_{w\alpha}} \operatorname{Tr} \rho_{\boldsymbol{\alpha}}^2$$

with $\rho_{\boldsymbol{\alpha}} = \operatorname{Tr}_{\boldsymbol{\alpha}^c} |\Psi\rangle\langle\Psi|$.

Computing concentratable entanglements

It follows then that

Concentratable entanglements

$$\begin{aligned} \mathcal{C}_{|\psi\rangle}(\boldsymbol{s}) &= 1 - \frac{1}{2^{c(\boldsymbol{s})}} \sum_{\alpha \in \mathcal{P}(\boldsymbol{s})} \operatorname{Tr} \rho_\alpha^2 \\ &= 1 - \sum_{\boldsymbol{z} \in \mathcal{Z}_0(\boldsymbol{s})} p(\boldsymbol{z}) \end{aligned}$$

- $\mathcal{Z}_0(s)$: set of bit strings with 0's on all idices in $\boldsymbol{s}$.

Entanglement concentration

- The action of this circuit on the input state $|\Psi\rangle = |\psi\rangle|\psi\rangle$ is

$$U_S^{\otimes n}|\mathbf{0}\rangle|\Psi\rangle = \frac{1}{2^n}\sum_{z=0}^{1}\prod_k |z_k\rangle(I_k + (-1)^{z_k}S_k)|\Psi\rangle$$

- At site j, the evolution of the two qubit system is given by

$$\rho_j \otimes \rho_j \rightarrow \sum_{\mu=\pm} K_{j,\mu}\,\rho_j \otimes \rho_j\,K_{j,\mu}^{\dagger}$$

with $K_{j,\mu} = \frac{1}{2}(I + \mu S_j)$.

K_+ and K_- project onto the ± 1 eigenspaces of S_j, with eigenstates

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$$

For each $|1\rangle$ in $|\boldsymbol{z}\rangle$ there is a singlet state.

Probabilistic Entanglement Concentration Protocol

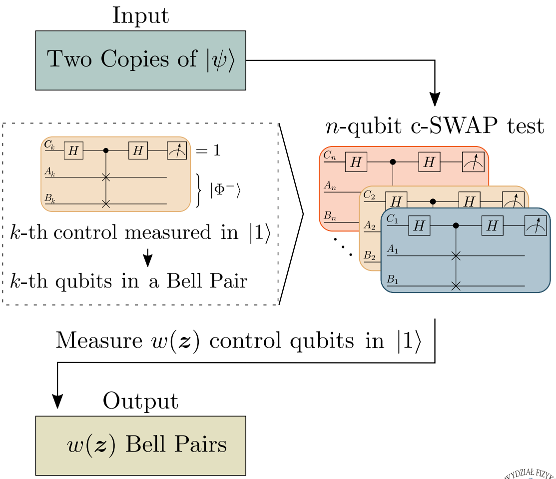


Table of Contents

W states: $|W\rangle = \frac{1}{\sqrt{n}} \sum_{\vec{x}:w(\vec{x})=1} |\vec{x}\rangle$

$p(\vec{z}) = \frac{1}{n^2}$ for all $\vec{z}$ with $w(\vec{z}) = 2$ and $p(\vec{z}) = 0$ for all $\vec{z}$ with $w(\vec{z}) > 2$, i.e., one can only have terms where $\vec{z}$ has only two ones.

$$\mathcal{C}_{|W\rangle}(s) = c(s)(2n - c(s) - 1)/2n^2.$$

GHZ states: $|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|\vec{0}\rangle + |\vec{1}\rangle\right)$

$p(\vec{z}) = \frac{1}{2^n}$ for all $\vec{z}$ with (even) Hamming weight $w(\vec{z}) \geq 2$

$$\mathcal{C}_{|GHZ\rangle}(s) = \frac{1}{2}\left(1 - 1/(2^{c(s)-\delta_{c(s)n}})\right)$$

GHZ vs W

Difference $\mathcal{C}_{|GHZ\rangle}(s) - \mathcal{C}_{|W\rangle}(s)$ vs number of qubits n for different sets s, with cardinalities $c(s) = 1, 2, n/2, n-1, n$.

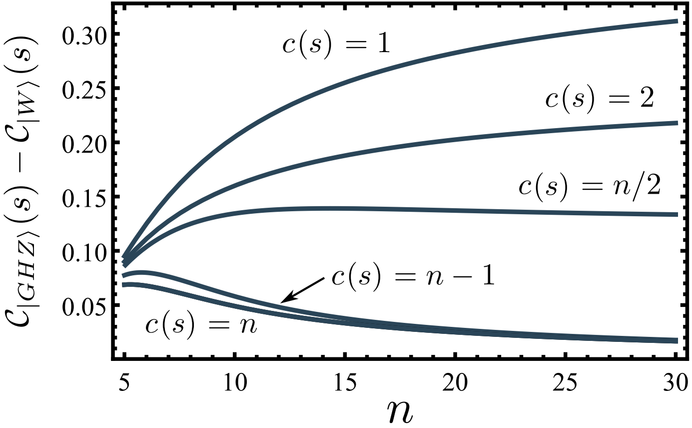


Table of Contents

Summing up

- **W**e defined a family of multipartite entanglement monotones that quantify entanglement in a multipartite system across every possible partition of the joint Hilber space. This quantities exhibit several properties that are desirable in a multipartite entanglement measure.
- **E**ach member of the family can be computed by means of a quantum cirquit implementing a parallelized controlled SWAP test, since their value is encoded in the probabilities of possible outcomes of a measurement in the control system.
- **I**n the case of W and GHZ states we have shown analytical expressions for computing the entanglement associated to any subset $\boldsymbol{s}$ as a function of its cardinality and the number of qubits.

Thank You!

Our work:

- Phys. Rev. Lett. 127, 140501 (2021).