UNIWERSYTET JAGIELLOŃSKI W KRAKOWIE

Chaos and Quantum Information Seminar
Jagiellonian University
Kraków

Computing Numerical and Exact SIC-POVMs

Markus Grassl

International Centre for Theory of Quantum Technologies
University Gdansk

markus.grassl@ug.edu.pl
sicpovm.markus-grassl.de

29 March 2021

in collaboration with
Marcus Appleby, Ingemar Bengtsson, Michael Harrison, Gary McConnell, Andrew Scott

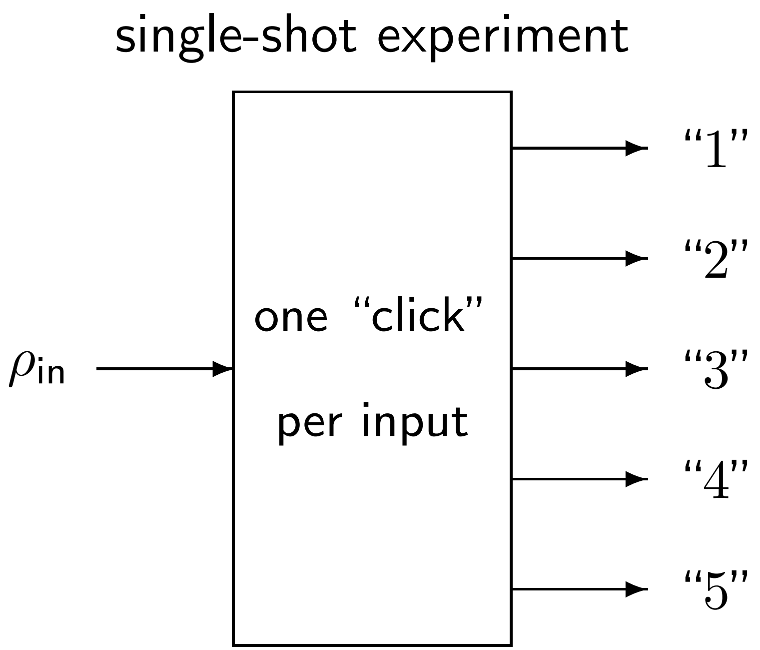
additional support by the Max Planck Institute for the Science of Light, Erlangen, and MPG

European Funds Smart Growth · Republic of Poland · FNP Foundation for Polish Science · European Union European Regional Development Fund

Positive Operator Valued Measures (POVMs)

General measurement device

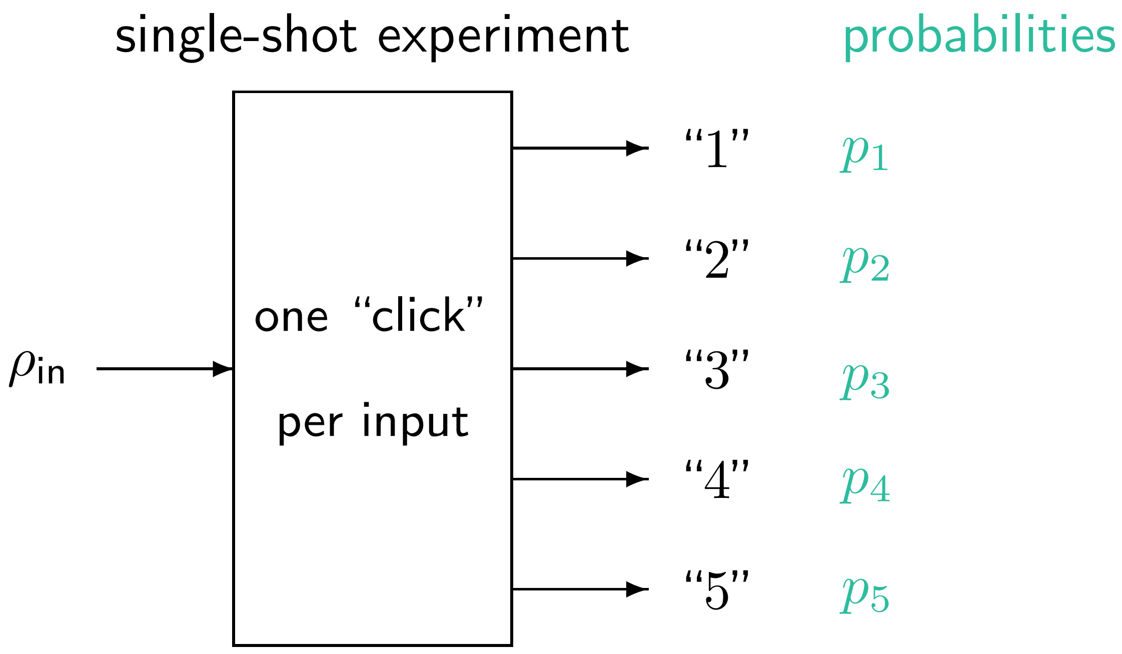
(from a quantum information processing point of view)



Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)

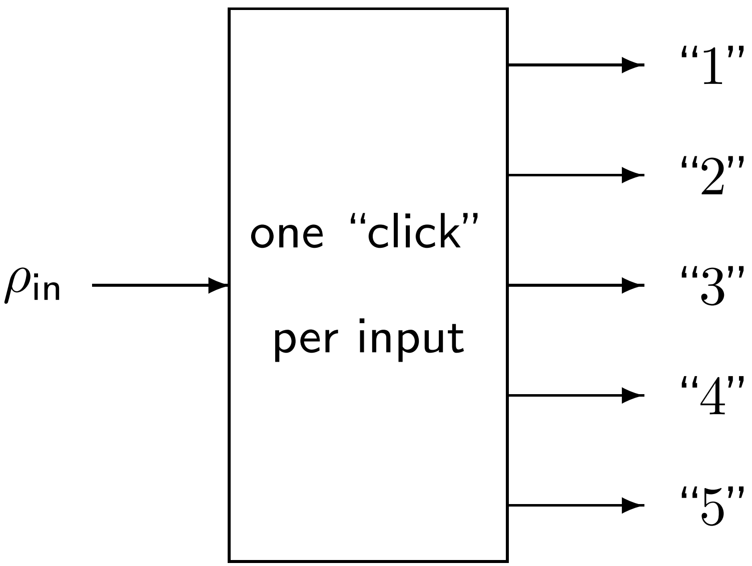


Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)

single-shot experiment



probabilities

$$p_1 = \mathrm{Tr}(\rho_{\text{in}}\Pi_1)$$

$$p_2 = \mathrm{Tr}(\rho_{\text{in}}\Pi_2)$$

$$p_3 = \mathrm{Tr}(\rho_{\text{in}}\Pi_3)$$

$$p_4 = \mathrm{Tr}(\rho_{\text{in}}\Pi_4)$$

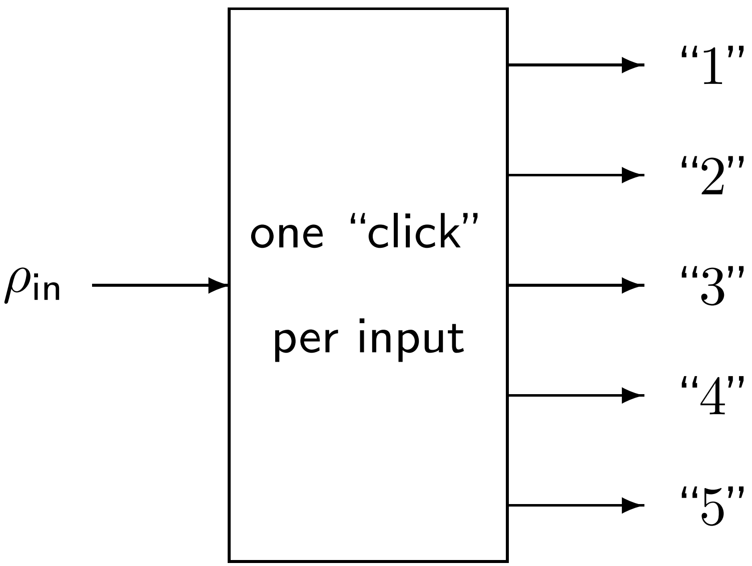
$$p_5 = \mathrm{Tr}(\rho_{\text{in}}\Pi_5)$$

Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)

single-shot experiment



probabilities

$$p_1 = \mathrm{Tr}(\rho_{\text{in}}\Pi_1) \geq 0$$

$$p_2 = \mathrm{Tr}(\rho_{\text{in}}\Pi_2) \geq 0$$

$$p_3 = \mathrm{Tr}(\rho_{\text{in}}\Pi_3) \geq 0$$

$$p_4 = \mathrm{Tr}(\rho_{\text{in}}\Pi_4) \geq 0$$

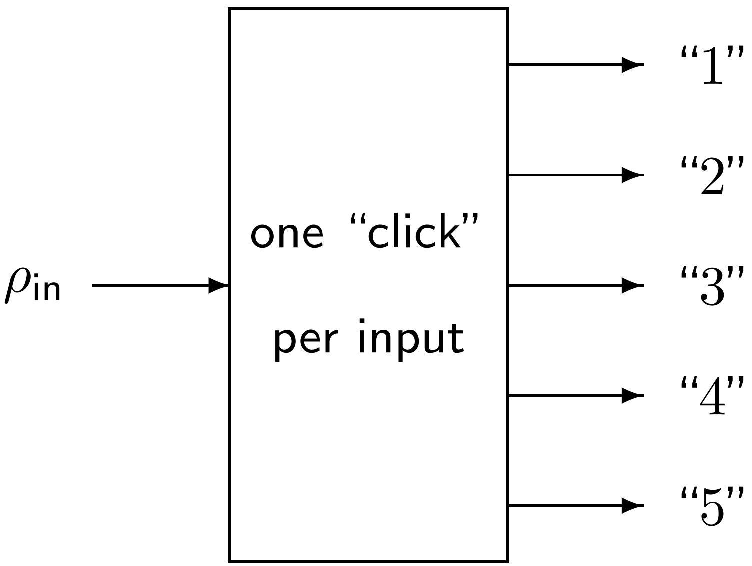
$$p_5 = \mathrm{Tr}(\rho_{\text{in}}\Pi_5) \geq 0$$

Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)

single-shot experiment



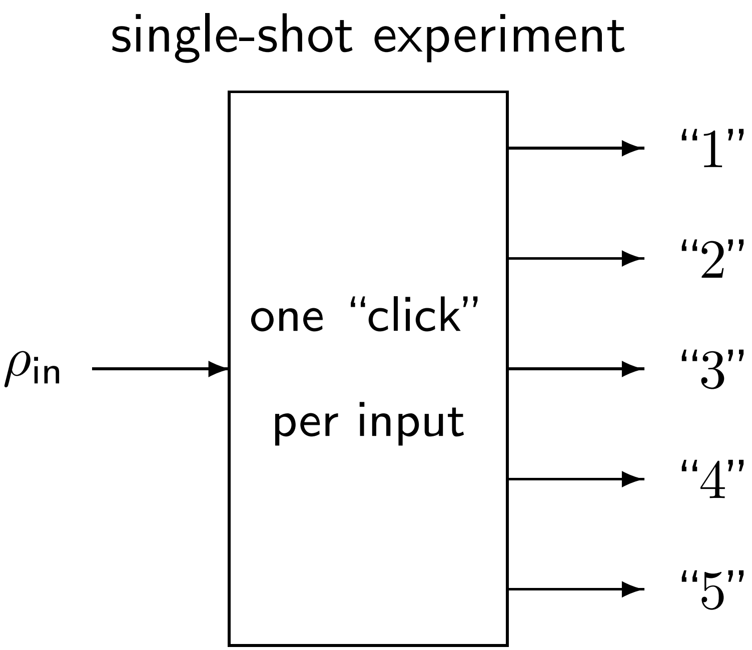
probabilities

$$\left.\begin{array}{l} p_1 = \mathrm{Tr}(\rho_{\text{in}}\Pi_1) \geq 0 \\ p_2 = \mathrm{Tr}(\rho_{\text{in}}\Pi_2) \geq 0 \\ p_3 = \mathrm{Tr}(\rho_{\text{in}}\Pi_3) \geq 0 \\ p_4 = \mathrm{Tr}(\rho_{\text{in}}\Pi_4) \geq 0 \\ p_5 = \mathrm{Tr}(\rho_{\text{in}}\Pi_5) \geq 0 \end{array}\right\} \Longrightarrow \Pi_i \geq 0$$

Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)



probabilities

$$\left.\begin{array}{l} p_1 = \mathrm{Tr}(\rho_{\text{in}}\Pi_1) \geq 0 \\ p_2 = \mathrm{Tr}(\rho_{\text{in}}\Pi_2) \geq 0 \\ p_3 = \mathrm{Tr}(\rho_{\text{in}}\Pi_3) \geq 0 \\ p_4 = \mathrm{Tr}(\rho_{\text{in}}\Pi_4) \geq 0 \\ p_5 = \mathrm{Tr}(\rho_{\text{in}}\Pi_5) \geq 0 \end{array}\right\} \Longrightarrow \Pi_i \geq 0$$

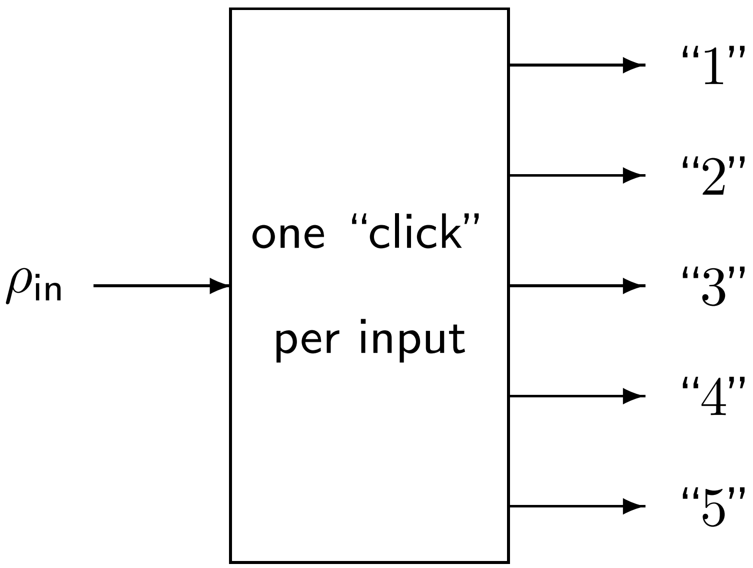
$$1 = \sum p_i = \mathrm{Tr}(\rho_{\text{in}} \sum \Pi_i)$$

Positive Operator Valued Measures (POVMs)

General measurement device

(from a quantum information processing point of view)

single-shot experiment



probabilities

$$p_1 = \operatorname{Tr}(\rho_{\text{in}}\Pi_1) \geq 0$$
$$p_2 = \operatorname{Tr}(\rho_{\text{in}}\Pi_2) \geq 0$$
$$p_3 = \operatorname{Tr}(\rho_{\text{in}}\Pi_3) \geq 0$$
$$p_4 = \operatorname{Tr}(\rho_{\text{in}}\Pi_4) \geq 0$$
$$p_5 = \operatorname{Tr}(\rho_{\text{in}}\Pi_5) \geq 0$$

$$\Longrightarrow \Pi_i \geq 0$$

$$1 = \sum p_i = \operatorname{Tr}(\rho_{\text{in}} \sum \Pi_i) \quad \Longrightarrow \sum \Pi_i = \mathbb{1}$$

Positive Operator Valued Measures (POVMs)

- set of m positive semidefinite operators Π_i with $\sum_{i=1}^{m} \Pi_i = \mathbb{1}$
- implementation with the help of an m-dimensional auxiliary system and a projective (von Neumann) measurement on the auxiliary system

$$\rho_{\text{in}} \otimes |0\rangle\langle 0| \longrightarrow U\left(\rho_{\text{in}} \otimes |0\rangle\langle 0|\right) U^\dagger \longrightarrow A_i \rho_{\text{in}} A_i^\dagger \otimes |i\rangle\langle i|$$

  with $A_i^\dagger A_i = \Pi_i$ (not unique), depending on U

  Neumark extension

- note: the number m of outcomes is independent of the dimension d of the Hilbert space of ρ_{in}

(Minimal) Informationally-Complete POVMs

- POVM with $m = d^2$ linearly independent operators Π_i
  $\Longrightarrow$ unique representation of any state as

$$\rho = \sum_{j=1}^{d^2} c_j \Pi_j$$

- measuring the POVM yields probabilities

$$p_i = \mathrm{Tr}(\rho\,\Pi_i) = \sum_{j=1}^{d^2} c_j\,\mathrm{Tr}(\Pi_i \Pi_j)$$

$\Longrightarrow$ reconstructing ρ by inverting the matrix with elements $\mathrm{Tr}(\Pi_i \Pi_j)$

Symmetric Informationally-Complete POVMs

- informationally complete POVM with rank-one elements $\Pi_i = \alpha|\psi_i\rangle\langle\psi_i|$
- symmetry: overlaps $\mathrm{tr}(\Pi_i\Pi_j) = \text{const} = \beta$ for $i \neq j$
- using $\sum \Pi_i = \mathbb{1}$ it follows that $\alpha = \frac{1}{d}$ and $\beta = \frac{1}{d^2(d+1)}$

$$\Longrightarrow |\langle\psi_i|\psi_j\rangle|^2 = \frac{1+\delta_{ij}d}{1+d}$$

- reconstruction

$$\rho = \sum_{j=1}^{d^2} \underbrace{\Big(d(d+1)\,\mathrm{Tr}(\rho\,\Pi_j) - 1\Big)}_{=c_j} \Pi_j$$

$\Longrightarrow$ coefficient c_j is an affine function of the probability $p_j = \mathrm{tr}(\rho\,\Pi_j)$

A Simple to State Problem

Are there d^2 vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d^2} \in \mathbb{C}^d$ in the complex vector space of dimension d such that:

(i) $\langle \boldsymbol{v}_j | \boldsymbol{v}_j \rangle = 1$ for $j = 1, \ldots, d^2$

(ii) $|\langle \boldsymbol{v}_j | \boldsymbol{v}_k \rangle|^2 = \dfrac{1}{d+1}$ for $1 \leq j < k \leq d^2$

The vectors $\boldsymbol{v}_j$ form an equiangular tight frame/finite unit norm tight frame.

A Simple to State Problem

Are there d^2 vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d^2} \in \mathbb{C}^d$ in the complex vector space of dimension d such that:

(i)	$\langle \boldsymbol{v}_j \| \boldsymbol{v}_j \rangle = 1$	for $j = 1, \ldots, d^2$
(ii)	$\|\langle \boldsymbol{v}_j \| \boldsymbol{v}_k \rangle\|^2 = \dfrac{1}{d+1}$	for $1 \leq j < k \leq d^2$

The vectors $\boldsymbol{v}_j$ form an equiangular tight frame/finite unit norm tight frame.

All solutions form a real algebraic variety, using $2d$ real variables per vector

$$\boldsymbol{v}_j = (a_{j1} + ib_{j1}, a_{j2} + ib_{j2}, \ldots, a_{j,d} + ib_{j,d})^T \qquad (i = \sqrt{-1})$$

$2d^3$ variables, d^2 equations (i) of degree 2 and $\binom{d^2}{2}$ equations (ii) of degree 4.

Weyl-Heisenberg Group

- generators: $H_d := \langle X, Z \rangle$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  $(\omega_d := \exp(2\pi i/d))$

- relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- basis:

$$H_d \big/ \zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Constructing SIC-POVMs

Ansatz:

SIC-POVM that is the orbit under the Weyl-Heisenberg group H_d, i. e.,

$$|\boldsymbol{v}^{(a,b)}\rangle := X^a Z^b |\boldsymbol{v}^{(0,0)}\rangle$$

$$|\langle \boldsymbol{v}^{(a,b)}|\boldsymbol{v}^{(a',b')}\rangle|^2 = \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases}$$

$$|\boldsymbol{v}^{(0,0)}\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |j\rangle,$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ we have to find only one *fiducial* vector $|\boldsymbol{v}^{(0,0)}\rangle$ instead of d^2 vectors

$\Longrightarrow$ polynomial equations with $2d-1$ variables, but already quite complicated for $d = 6$

Jacobi Group (or Clifford Group)

- automorphism group of the Weyl-Heisenberg group H_d, i. e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $\mathrm{SL}(2, \mathbb{Z}_d)$, i. e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix}, \ \tilde{T} \in \mathrm{SL}(2, \mathbb{Z}_d)$$

$\Longrightarrow$ homomorphism $J_d \to \mathrm{SL}(2, \mathbb{Z}_d)$

- additionally: complex conjugation (anti-unitary)

$$X^a Z^b \mapsto X^a Z^{-b} \qquad \text{corresponding to } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

Conjecture:

For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator E_0 under the Weyl-Heisenberg group H_d.
What is more, E_0 commutes with an element S of the Jacobi group J_d.
The action of S on H_d modulo the center has order three.

support for this conjecture (to date):

- numerical solutions for all dimensions $d \leq 193$, plus a few more
- exact algebraic solutions for some dimensions (see below)

one of the prize problems in
[Paweł Horodecki, Łukasz Rudnicki, Karol Życzkowski, Five open problems in quantum information, arXiv:2002.03233]

Numerical Search for SIC-POVMs

- for any state $|\psi\rangle \in \mathbb{C}^d$

$$f(|\psi\rangle) = \sum_{j,k=1}^{d} \Big| \sum_{\ell=1}^{d} \langle\psi|j+\ell\rangle\langle\ell|\psi\rangle\langle\psi|k+\ell\rangle\langle j+k+\ell|\psi\rangle \Big|^2$$

$$= \sum_{j,k=1}^{d} \Big| \sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell} \Big|^2 \qquad \geq \frac{2}{d+1}$$

  with equality iff $|\psi\rangle$ is a fiducial vector for a Weyl-Heisenberg SIC-POVM

- gradient descent to minimize $f(|\psi\rangle)$, subject to unit norm

Numerical Search for SIC-POVMs

- for any state $|\psi\rangle \in \mathbb{C}^d$

$$f(|\psi\rangle) = \sum_{j,k=1}^{d} \Big| \sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell} \Big|^2 \geq \frac{2}{d+1}$$

  with equality iff $|\psi\rangle$ is a fiducial vector for a Weyl-Heisenberg SIC-POVM

- gradient descent to minimize $f(|\psi\rangle)$, subject to unit norm

- use $F(\vec{x}) = f\left(\frac{P\vec{x}}{\|P\vec{x}\|}\right)$ for an arbitrary vector $\vec{x} \in \mathbb{C}^d$, where P is the projection onto a subspace (prescribed symmetry)

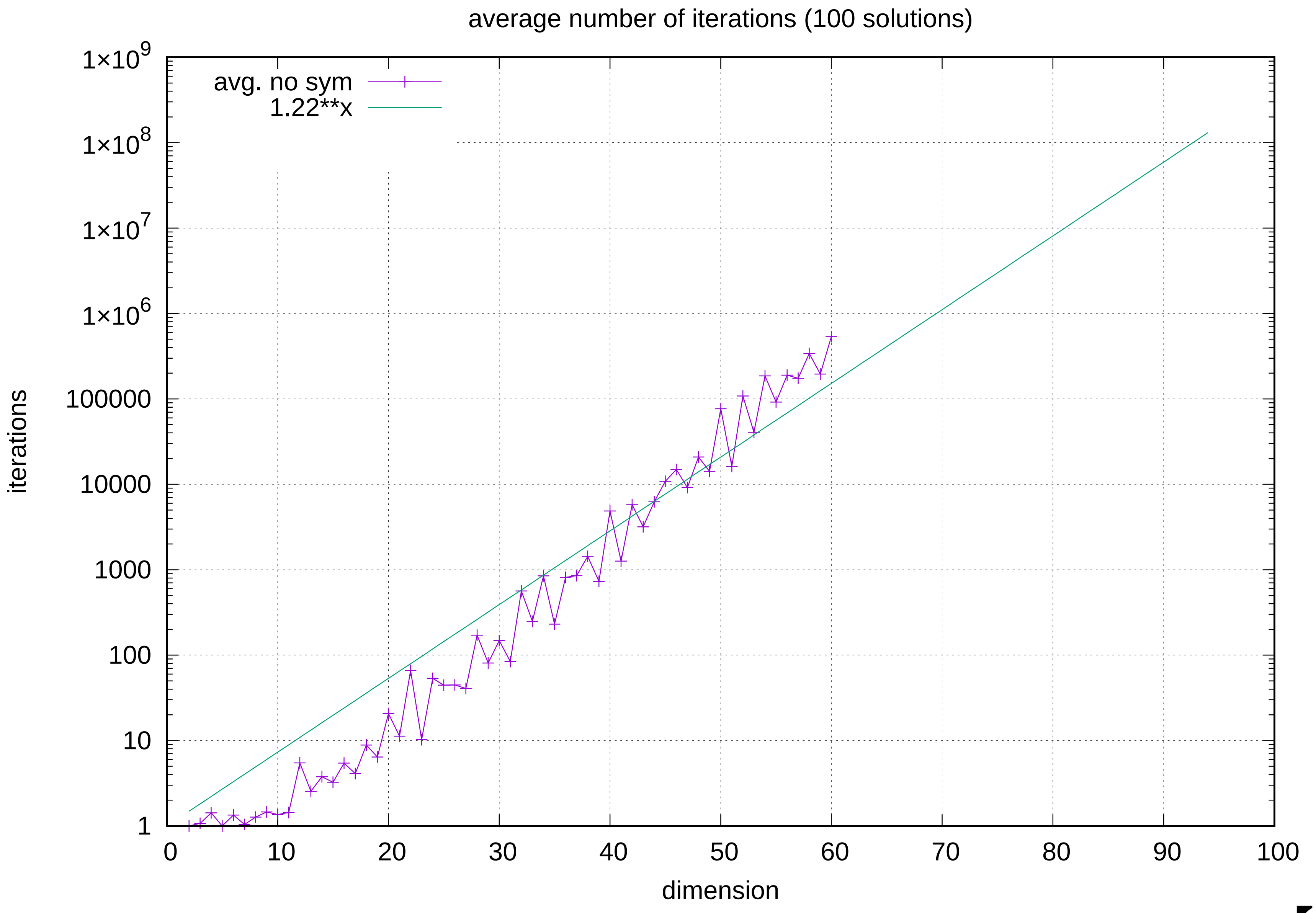
- chain rule yields a relatively simple formula for the gradient of $F(\vec{x})$ in terms of the gradient of f

- complexity $O(d^3)$ for both the function and the gradient when storing $O(d^2)$ intermediate values

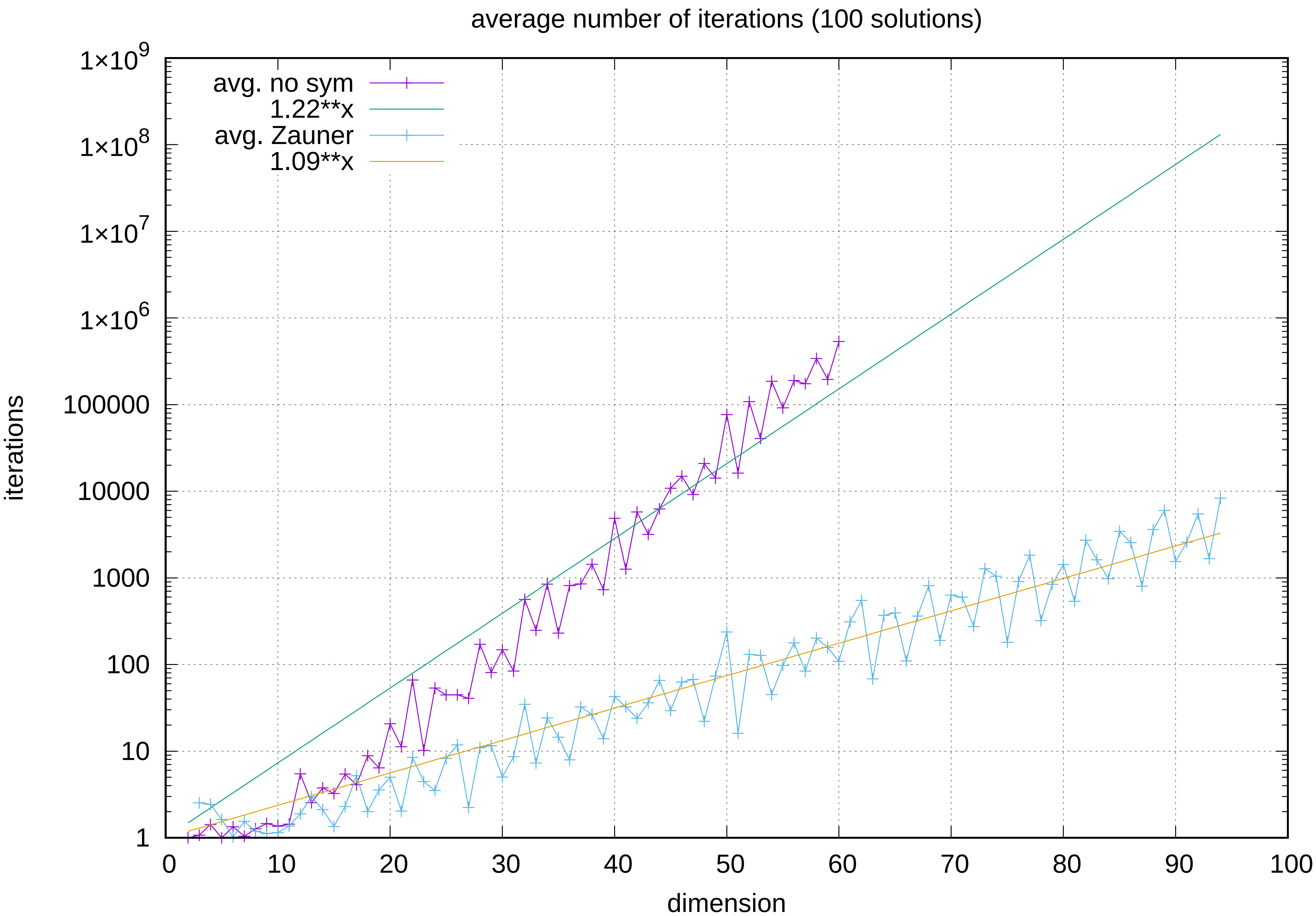
Numerical Search for SIC-POVMs

- efficient implementation of $F(\vec{x})$ and its gradient in `C++` by Andrew Scott
- parallel computation of the function/gradient using OpenMP/CUDA
- minimization using limited-memory Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm
- search runs into local minima, we need many random initial points
- running many instances on HPC clusters by MPG and GWDG
-
  - for $d = 189$: approx. 23.3×10^6 trials, 3.48 CPU years
  - for $d = 190$: approx. 66.8×10^6 trials, 10.51 CPU years
  - for $d = 193$: approx. 78.3×10^6 trials, 13.00 CPU years
  - for $d = 5779$: 55065 trials, 17.69 GPU years, no success

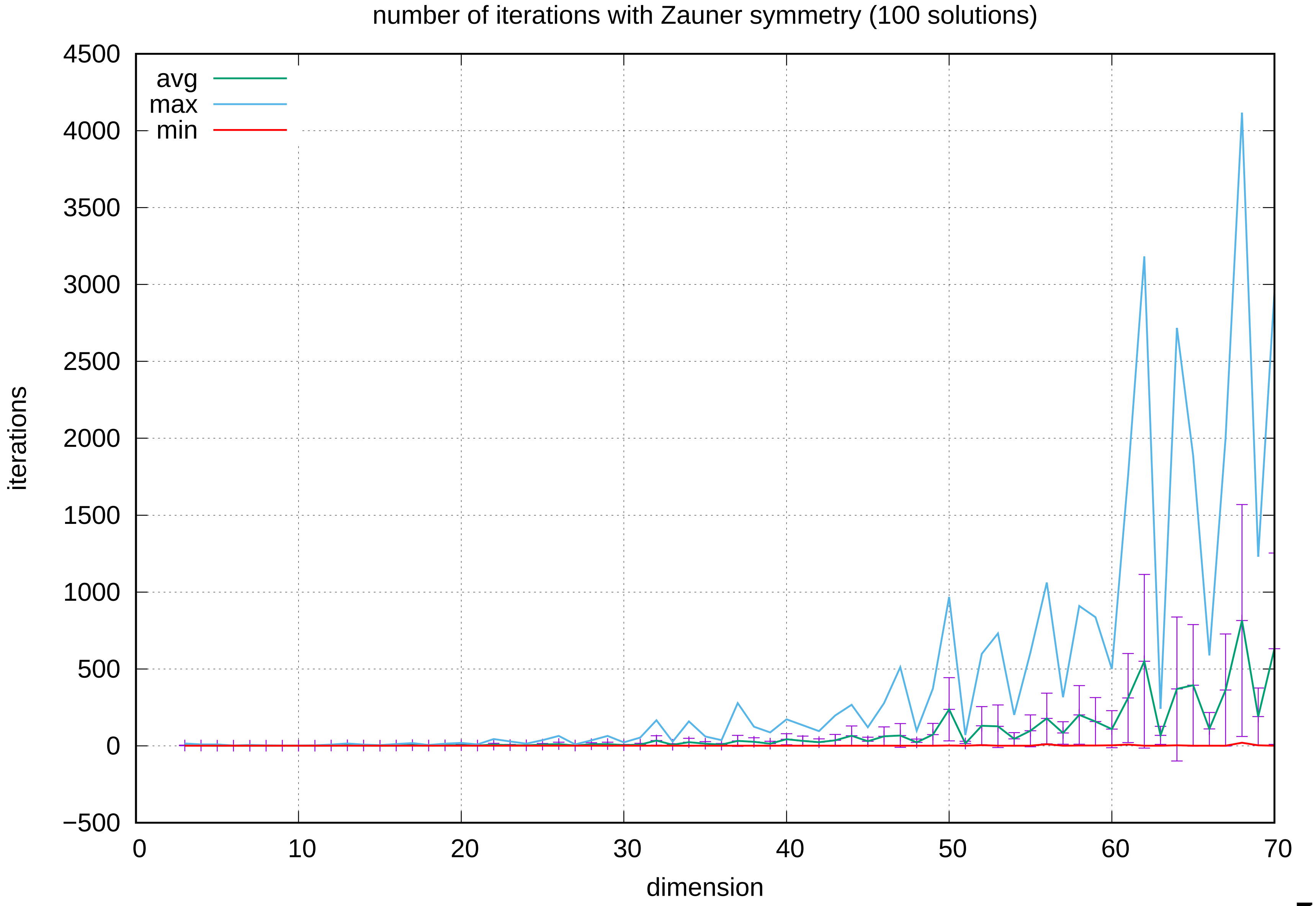
Average Number of Iterations



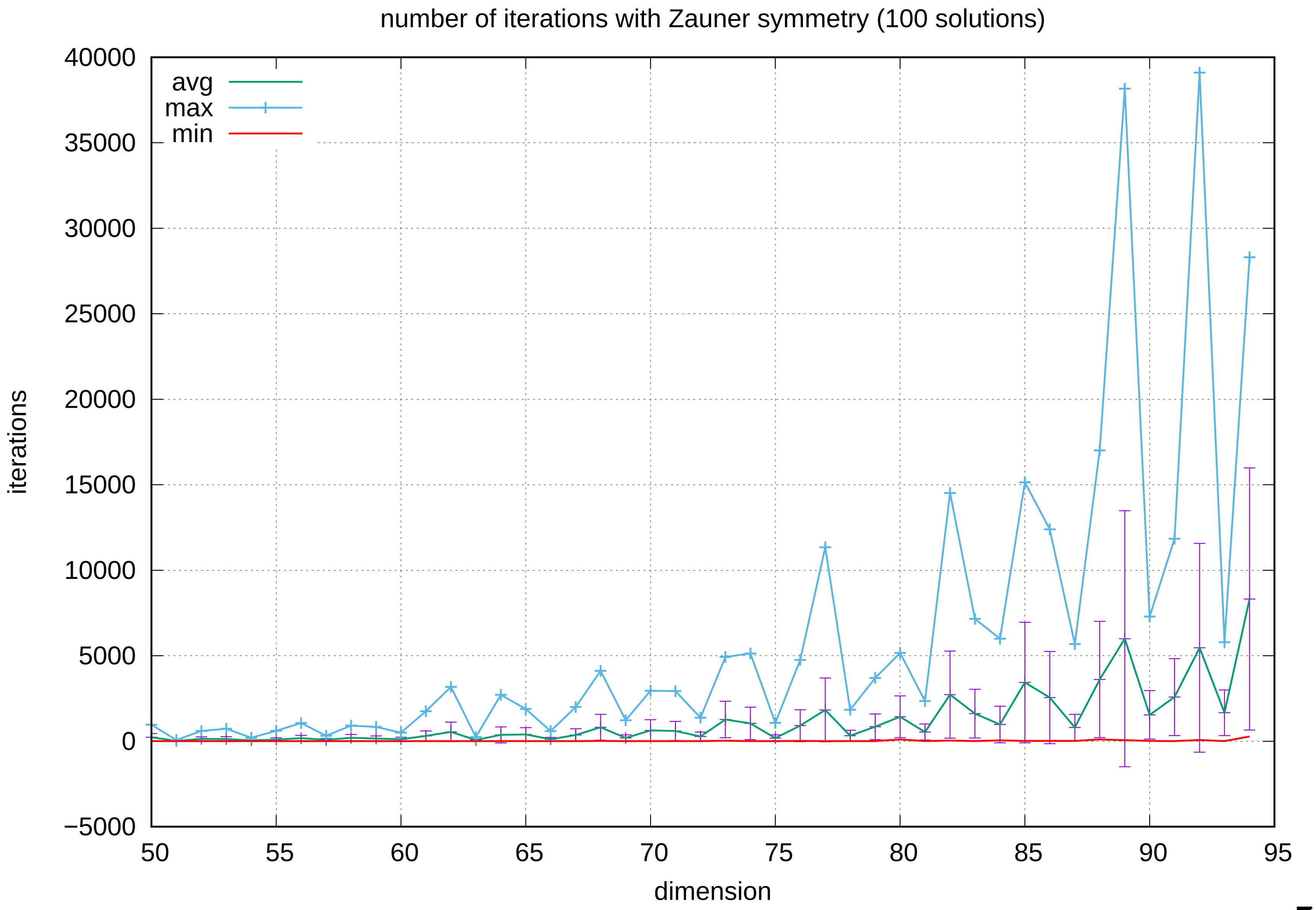
Average Number of Iterations



Average Number of Iterations with Zauner Symmetry



Average Number of Iterations with Zauner Symmetry



Fibonacci-Lucas SIC-POVMs

[Markus Grassl & Andrew J. Scott, JMP 58, December 2017, arXiv:1707.02944]

- (exact) symmetry analysis of a numerical solution for $d = 124$
  $\Longrightarrow$ symmetry group of order 30 (prescribed order 6)
- identified as part of a series of dimensions (related to Lucas numbers)
  $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128, 39604, \ldots$
- symmetry group of order $6k$ related to Fibonacci numbers
- new exact solutions for $d = 124$ and $d = 323$ (previously $d = 48$)
- new numerical solutions for $d = 844$ and $d = 2208$ (previously $d = 323$)

Fibonacci-Lucas SIC-POVMs

[Markus Grassl & Andrew J. Scott, JMP 58, December 2017, arXiv:1707.02944]

- Fibonacci numbers F_k with $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$
- Lucas numbers L_k with $L_0 = 2$, $L_1 = 1$, $L_{k+1} = L_k + L_{k-1}$
- prescribed anti-unitary symmetry related to the Fibonacci matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

- modulo $d_k = L_{2k} + 1$, the matrix A has order $6k$
- sequence of dimensions
  $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128, 39604, \ldots$
- squarefree part D of $(d+1)(d-3)$ is always $D = 5$
  $\Longrightarrow$ ray class field over $\mathbb{Q}(\sqrt{5})$

Generalised Fibonacci-Lucas SIC-POVMs

- generalised Fibonacci numbers $F_{m,k}$ with
  $F_{m,0} = 0$, $F_{m,1} = 1$, $F_{m,k+1} = mF_{m,k} + F_{m,k-1}$
- generalised Lucas numbers $L_{m,k}$ with
  $L_{m,0} = 2$, $L_{m,1} = m$, $L_{m,k+1} = mL_{m,k} + L_{m,k-1}$
- prescribed anti-unitary symmetry related to the matrix

$$A_m = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \qquad A_m^k = \begin{pmatrix} F_{m,k-1} & F_{m,k} \\ F_{m,k} & F_{m,k+1} \end{pmatrix}$$

- modulo $d_{m,k} = L_{m,2k} + 1$, the matrix A_m has order $6k$
- squarefree part D of $(d+1)(d-3)$ equals the squarefree part of $m^2 + 4$
  $\Longrightarrow$ ray class field over $\mathbb{Q}(\sqrt{D})$

Dimensions with Anti-Unitary Symmetries

	k	1	2	3	4	5	6	7	8
	ord(F)	6	12	18	24	30	36	42	48
m	D	$F_{e'}$	F_g						
1	5	4	8	19	48	124	323	844	2208
2	2	7	35	199	1155	6727	39203	228487	1331715
3	13	12	120	1299	14160	154452	1684803	18378372	200477280
4	5	19	323	5779	103683	1860499			
5	29	28	728	19603	528528	14250628			
6	10	39	1443	54759	2079363	78960999			
7	53	52	2600	132499	6754800				
8	17	67	4355	287299	18957315				
9	85	84	6888	571539	47430768				
10	26	103	10403	1060903					
11	5	124	15128	1860499					
12	37	147	21315	3111699					
13	173	172	29240	4999699					
14	2	199	39203	7761799					
15	229	228	51528						
16	65	259	66563						
17	293	292	84680						
18	82	327	106275						
19	365	364	131768						
20	101	403	161603						

Families of SIC-POVMs with Unitary Symmetry

- prescribed unitary symmetry related to the matrix

$$B_m = \begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix}$$

- similar recurrence relations for the entries of B_m^k and the corresponding dimension
- order of the symmetry is $3k$

Dimensions with Unitary Symmetries

	k	1	2	3	4	5	6	7	8	9
	ord(F)	3	6	9	12	15	18	21	24	27
m	D	F_z	F_b	F_d						
3	5	*4*	*8*	*19*	*48*	*124*	*323*	*844*	*2208*	*5779*
4	3	5	15	53	195	725	2703	10085	37635	140453
5	21	6	24	111	528	2526	12099	57966	277728	1330671
6	2	7	35	199	*1155*	6727	39203	228487	1331715	7761799
7	5	8	48	323	*2208*	15128	103683	710648	4870848	33385283
8	15	9	63	489	3843	30249	238143	1874889	14760963	116212809
9	77	10	80	703	6240	55450	492803	4379770	38925120	345946303
10	6	11	99	971	9603	95051	940899	9313931	92198403	912670091
11	13	12	120	*1299*	14160	154452				
12	35	13	143	1693	20163	240253				
13	165	14	168	2159	27888	360374				
14	3	15	195	2703	37635	524175				
15	221	16	224	3331	49728	742576				
16	7	17	255	4049	64515					
17	285	18	288	4863	82368					
18	5	19	*323*	5779	103683					
19	357	20	360	6803	128880					
20	11	21	399	7941	158403					

Symmetries and Ray Class Fields

[Appleby, Chien, Flammia & Waldron, J. Phys. A. 51, 2018, arXiv:1703.05981]

Ray class field conjecture

nested tower of fields (for the minimal field)

$$\mathbb{Q} \lhd \mathbb{E}_c = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E} = \mathbb{E}_1(i\sqrt{d'}).$$

- $\mathbb{E}$ is the ray class field over $\mathbb{Q}(\sqrt{D})$ with conductor d' with ramification at both infinite places, $a^2 D = (d+1)(d-3)$
- $\mathbb{E}_1$ is the ray class field with ramification only allowed at the infinite place taking $\sqrt{D}$ to a positive real number
- $\mathbb{E}_0$ is the Hilbert class field over $\mathbb{Q}(\sqrt{D})$, in particular $[\mathbb{E}_0 : \mathbb{Q}(\sqrt{D})]$ equals the class number of $\mathbb{Q}(\sqrt{D})$

Symmetries and Ray Class Fields

[Appleby, Chien, Flammia & Waldron, J. Phys. A. 51, 2018, arXiv:1703.05981]

Ray class field conjecture

nested tower of fields (for the minimal field)

$$\mathbb{Q} \lhd \mathbb{E}_c = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E} = \mathbb{E}_1(i\sqrt{d'}).$$

- for $\mathcal{M}$ a certain maximal Abelian subgroup of $\mathrm{GL}(2, \mathbb{Z}/d'\mathbb{Z})$ and (essentially) the symmetry group $S(\Pi)$ of the SIC-POVM:

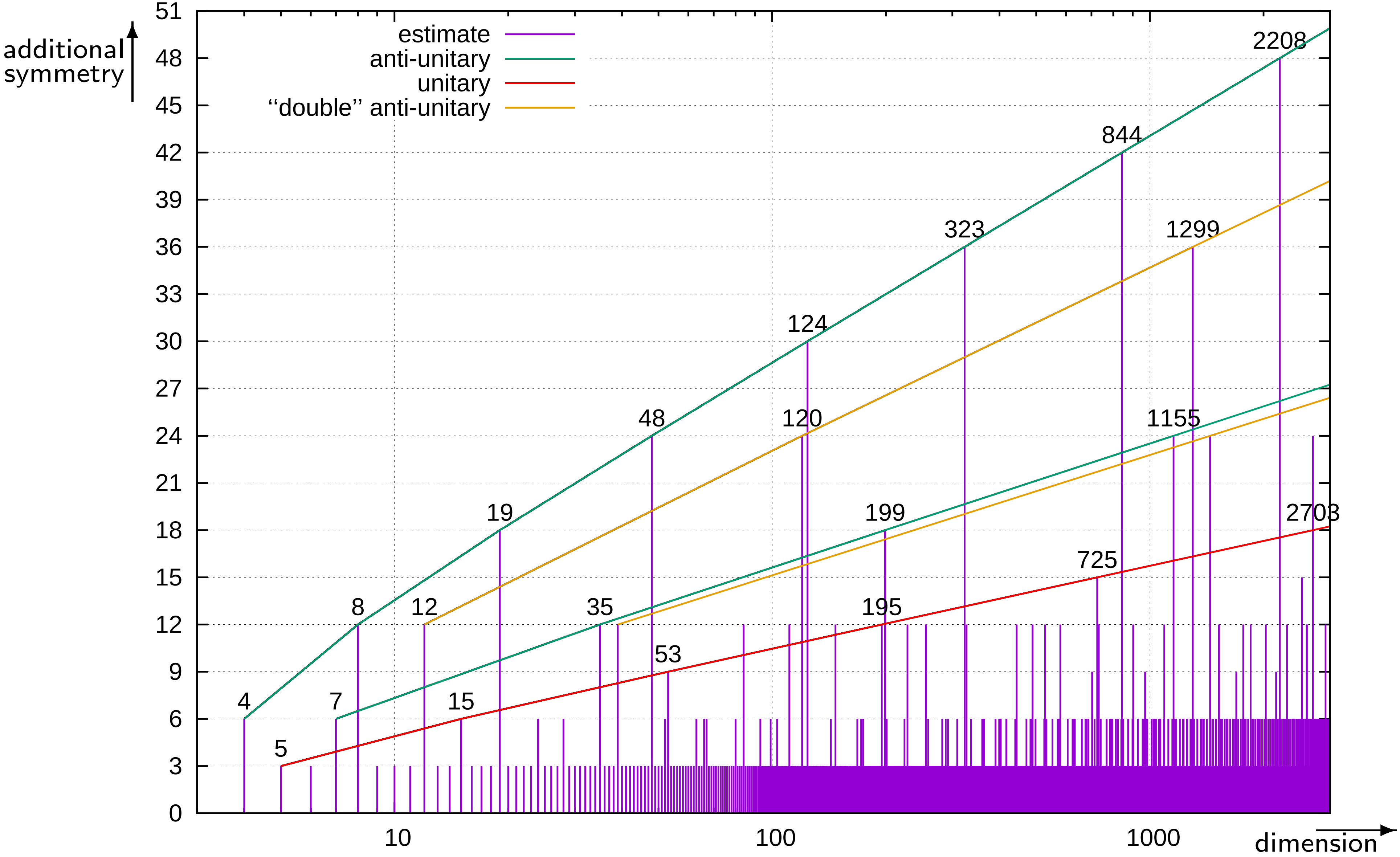
$$\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0) \cong \mathcal{M}/S(\Pi)$$

our observation: estimate for the group order:

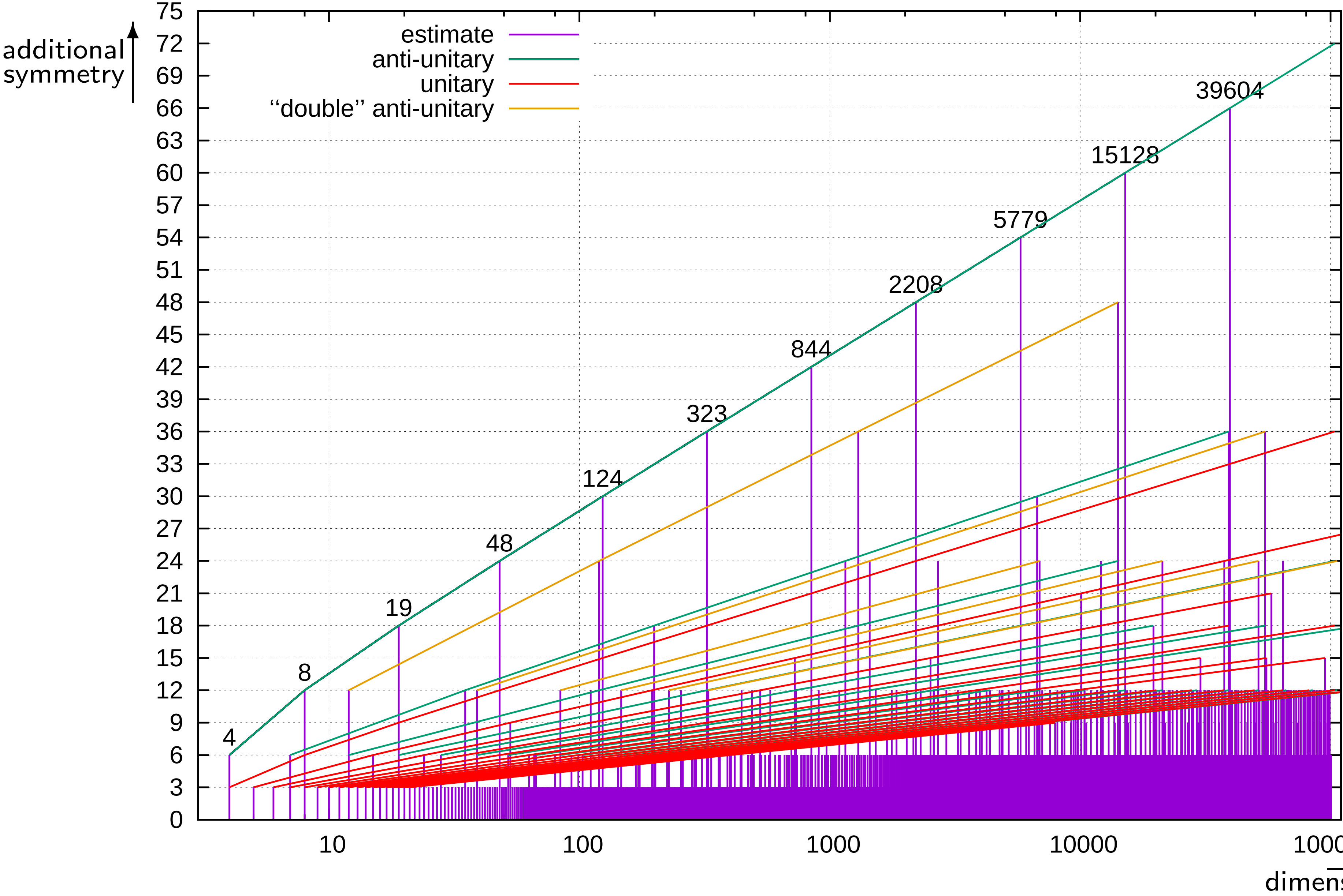
$$|S(\Pi)| = \frac{|\mathcal{M}|}{|\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)|} = \frac{|\mathcal{M}| \times |\mathrm{Gal}(\mathbb{E}_0/\mathbb{E}_c)|}{|\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_c)|}$$

- 1 or 4 cases for $|\mathcal{M}|$, but $|S(\Pi)|$ must be integral

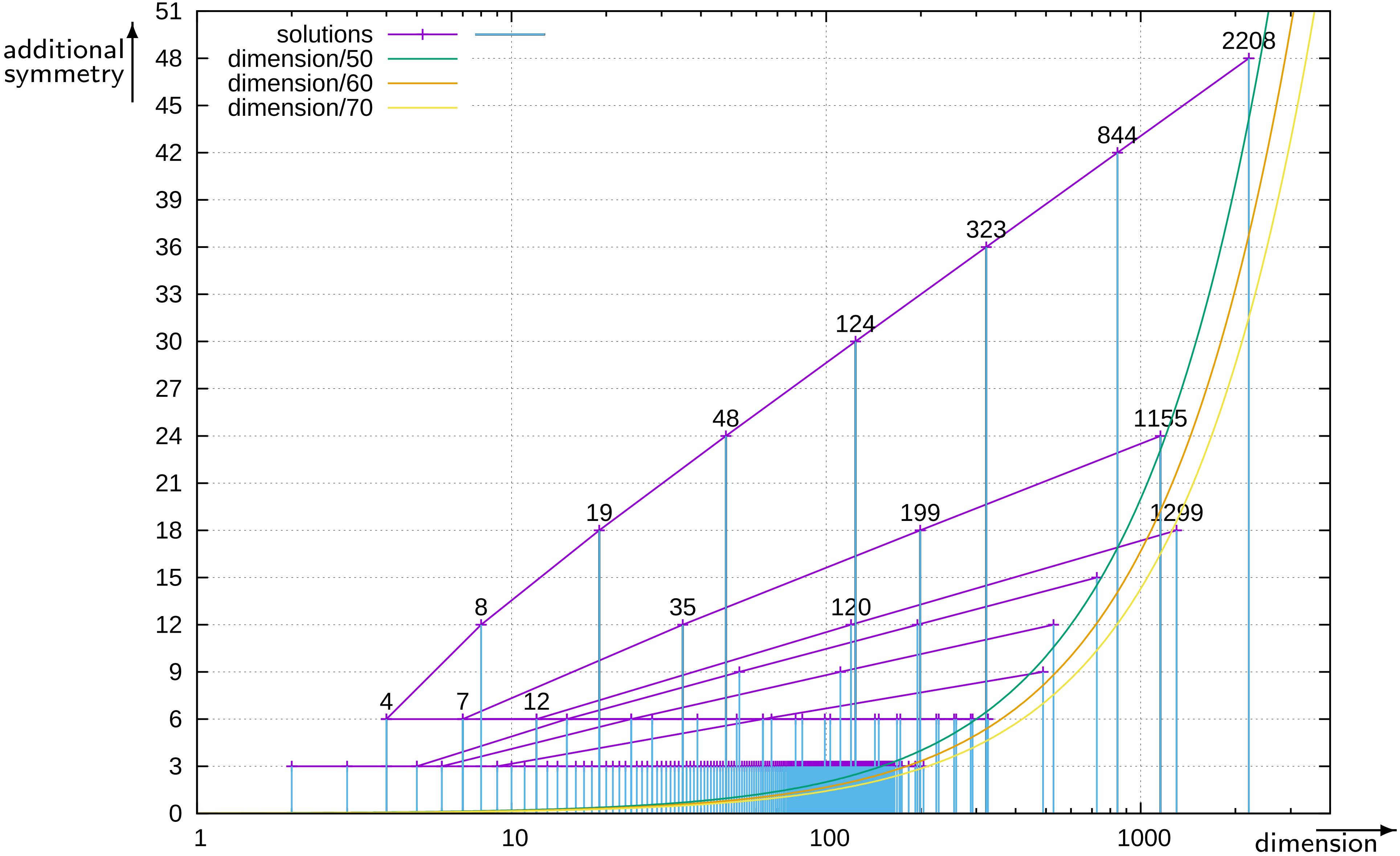
Symmetries of SIC-POVMs



Symmetries of SIC-POVMs



Known SIC-POVMs



Exact Solutions from Numerical Solutions

[Appleby, Chien, Flammia & Waldron, J. Phys. A. 51, 2018, arXiv:1703.05981]

- ray class field conjecture: $\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0) \cong \mathcal{M}/S(\Pi)$
- for $\sigma \in \mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)$, we have a matrix $G_\sigma \in \mathrm{GL}(2, \mathbb{Z}/d'\mathbb{Z})$ such that

$$\sigma\left(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\right) = \mathrm{Tr}(\Pi D_{G_\sigma \boldsymbol{p}})$$

- expansion coefficients $c_{\boldsymbol{p}} = \mathrm{Tr}(\Pi D_{\boldsymbol{p}})$ in the same orbit under $\mathcal{M}$ are related by Galois conjugation
- the coefficients of the polynomial $f_{\boldsymbol{p}_0}(z) = \prod_{\boldsymbol{p} \in \boldsymbol{p}_0^{\mathcal{M}}} (z - c_{\boldsymbol{p}})$

  lie in a number field of "small" degree
- find the exact minimal polynomials of those coefficients (requires high-precision numerical solution)
- find the roots of the exact polynomials $f_{\boldsymbol{p}_0}(z)$ in the ray class field

More Exact Solutions from Numerical Solutions

[Markus Grassl, arXiv:210?.????]

- when G_σ has determinant 1:

$$\sigma\left(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\right) = \mathrm{Tr}(\Pi D_{G_\sigma \boldsymbol{p}}) = \mathrm{Tr}(\Pi T_\sigma D_{\boldsymbol{p}} T_\sigma^\dagger) = \mathrm{Tr}(T_\sigma^\dagger \Pi T_\sigma D_{\boldsymbol{p}})$$

  $\Longrightarrow$ action of $T_\sigma^\dagger$ on the projection Π and the state $|\psi\rangle$

- when T_σ is a permutation matrix, we obtain a Galois action on the cofficients of the state $|\psi\rangle$
  $\Longrightarrow$ similar approach for polynomials with the coefficients of $|\psi\rangle$ as roots

- new exact solutions for 57 additional dimensions (so far)

$$d = 26, 38, 42, 49, 52, 57, 61, 62, 63, 65, 67, 73, 74, 78, 79, 84, 86, 91, 93, 95, 97, 98, 103, 109, 111, 122, 127, 129, 133, 134, 139, 143, 146, 147, 151, 155, 157, 163, 168, 169, 172, 181, 182, 183, 193, 199, 201, 228, 259, 292, 327, 364, 399, 403, 489, 844, 1299$$

- fiducial vectors lie in a proper ("small") subfield of the ray class field

Direct Conversion of the Fiducial Vector

[M. Appleby & I. Bengtsson, Simplified exact SICs, JMP 60, June 2019, arXiv:1811.00947]

- start with a numerical fiducial vector
- instead of the standard basis (of the Weyl-Heisenberg group), express the vector with respect to an adapted basis of the eigenspace of the symmetry
- for some dimensions, the absolute values and phases/ratios of phases of the expansion coefficients have "nice" exact minimal polynomials
- demonstrated for $d = 5, 15, 195$ (with $D = 3$) by Appleby & Bengtsson
- additional exact solution for $d = 725$ (also with $D = 3$), final result in number field of degree $134\,400$
- difficulties
  - additional square roots are required for intermediate results
  - so far unclear for which dimensions this will work

Hot Off the Fire

new recipes under development with Marcus Appleby, Ingemar Bengtsson, Michael Harrison, Gary McConnell

- analysing exact solutions in certain dimensions led to new conjectures relating SIC-POVMs and number theory
- does not require a numerical fiducial vector
- avoids factoring exact polynomials over number fields of large degree
- intermediate steps require (so far) high precision
- new solutions:
  - $d = 5779$ (2021-02-12)
  - $d = 787$ (2021-02-12)
  - $d = 487$ (2021-02-18)

Current State of the SIC-POVM Problem

- **numerical solutions:**
  - $d \leq 45$: Joesph Renes et al. quant-ph/0310075
  - $d \leq 67$: Andrew Scott & Markus Grassl arXiv:0910.5784
  - $d \leq 121$ plus a few more: Andrew Scott arXiv:1703.03993
  - $d \leq 151$: Christopher Fuchs et al. arXiv:1703.07901
  - $d \leq 193$, $d = 204, 224, 255, 288, 528, 1155, 2208$
- **exact algebraic solutions:**
  - early 2017: $d = 2$–$24, 28, 30, 31, 35, 37, 39, 43, 48, 124$ (32 dimensions)
  - all dimensions $d \leq 53, 57, 61, 62, 63, 65, 67, 73, 74, 76, 78, 79, 80, 84, 86, 91, 93, 95, 97, 98, 99, 103, 109, 111, 120, 122, 124, 127, 129, 133, 134, 139, 143, 146, 147, 151, 155, 157, 163, 168, 169, 172, 181, 182, 183, 193, 195, 199, 201, 228, 259, 292, 323, 327, 364, 399, 403,$ *487*, $489, 725,$ *787*, $844, 1299,$ *5779* (115 dimensions)

Thank you!
Danke! Merci!
Dziekuje!

Acknowledgments

The ‘International Centre for Theory of Quantum Technologies’ project (contract no. 2018/MAB/5) is carried out within the International Research Agendas Programme of the Foundation for Polish Science co-financed by the European Union from the funds of the Smart Growth Operational Programme, axis IV: Increasing the research potential (Measure 4.3).

European Funds Smart Growth | Republic of Poland | FNP Foundation for Polish Science | European Union European Regional Development Fund

Additional computing resources were provided via MPL MAX PLANCK INSTITUTE FOR THE SCIENCE OF LIGHT