Unambiguous quantum error correction codes on mixed states

Ryszard Kukulski, Łukasz Pawela, Zbigniew Puchała

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Quantum error correction

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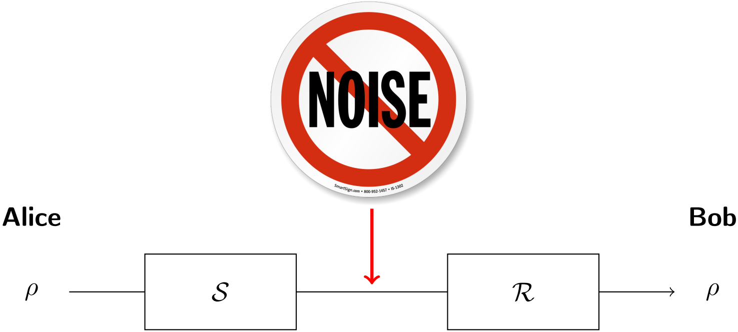
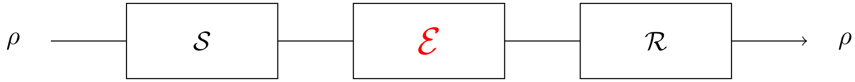


What now?

Let $\mathcal{E} \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ be the considered noise channel. $\left(\mathcal{E} \leftarrow \int_{\xi} \mathcal{E}\mu(d\mathcal{E})\right)$



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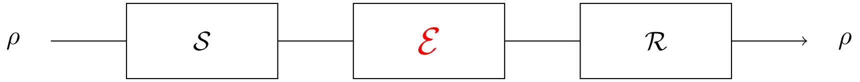

Goal

Find $\mathcal{S} \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ and $\mathcal{R} \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ such that $\mathcal{R}\mathcal{E}\mathcal{S}(\rho) = \rho$.

Quantum codes, Knill-Laflamme Theorem

Let $C \subset \mathcal{Y}$ be a code subspace and P_C be a projection on the subspace C.

Definition

The code C is correctable for $\mathcal{E}$ if there exists $\mathcal{R} \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ such that

$$\mathcal{R}\mathcal{E}\mathcal{S} = \mathcal{S}, \tag{1}$$

where $\mathcal{S}(Y) = P_C Y P_C$.

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Theorem Knill-Laflamme[1]

Assume that $\mathcal{E} = \{\mathcal{E}_i\}$ is a set of Kraus operators of a channel $\mathcal{E}$. The code C is correctable for $\mathcal{E}$ if and only if there exists matrix α such that

$$P_C \mathcal{E}_j^\dagger \mathcal{E}_i P_C = \alpha_{ji} P_C. \tag{2}$$

[1] Knill, Emanuel, and Raymond Laflamme. "Theory of quantum error-correcting codes." Physical Review A 55.2 (1997): 900.

Standardization

Remark

Assume that there exist $\mathcal{S} = P_C \cdot P_C \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ and $\mathcal{R} \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$ such that

$$\mathcal{R}\mathcal{E}\mathcal{S} = \mathcal{S}. \tag{3}$$

Let $\mathcal{X} = \mathbb{C}^d$, where $d = \mathrm{rank}(P_C) \leq \dim(\mathcal{Y})$.
Then, there exist $\tilde{\mathcal{S}} \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ and $\tilde{\mathcal{R}} \in \mathcal{C}(\mathcal{Y}, \mathcal{X})$ such that

$$\tilde{\mathcal{R}}\mathcal{E}\tilde{\mathcal{S}} = \mathcal{I}_{\mathcal{X}}. \tag{4}$$

Probabilistic quantum computation

Subchannels

An operation $\mathcal{S}$ is subchannel ($\mathcal{S} \in s\mathcal{C}(\mathcal{X}, \mathcal{Y})$) if:

- $\mathcal{S}$ is completely positive (CP)
- $\mathcal{S}$ is trace non-increasing *i.e.* $\operatorname{tr} \mathcal{S}(\rho) \leq 1, \ \forall \rho \in \mathcal{D}(\mathcal{X})$

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Let $J(\mathcal{S})$ be a Choi-Jamiołkowski matrix of $\mathcal{S}$ defined as
$J(\mathcal{S}) \coloneqq (\mathcal{S} \otimes \mathcal{I}_{\mathcal{X}})(|\mathbb{1}_{\mathcal{X}}\rangle\rangle\langle\langle\mathbb{1}_{\mathcal{X}}|)$.

Property

$\mathcal{S}$ is trace non-increasing if and only if $\mathrm{tr}_1(J(\mathcal{S})) \leq \mathbb{1}_{\mathcal{X}}$.

Subchannels realization

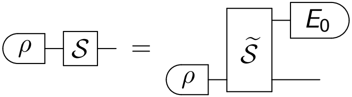
Realization of subchannel $\mathcal{S}$

Let us define a channel $\widetilde{\mathcal{S}} \in \mathcal{C}(\mathcal{X}, \mathbb{C}^2 \otimes \mathcal{Y})$ as

$$\widetilde{\mathcal{S}}(X) = |0\rangle\langle 0| \otimes \mathcal{S}(X) + |1\rangle\langle 1| \otimes \Psi(X), \tag{5}$$

where $\mathcal{S}$ and Ψ be subchannels such that $(\mathcal{S} + \Psi) \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$.

A realization of a subchannel $\mathcal{S}$ acts as



(6)

Figure: Realization of a subchannel $\mathcal{S}$ by using measurement $E = \{E_0, E_1\}$, where $E_i = |i\rangle\langle i|$, $i = 0, 1$.

Unambiguous quantum error correction

We considered

$$\mathcal{RES}(\rho) = \rho.$$

How much more noise we are able to correct if we assume $\mathcal{RES}(\rho) \propto \rho$?

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Definition

The quantum information embedded in $\mathcal{X}$ is unambiguously correctable for noise channel $\mathcal{E}$ if there exist subchannels $\mathcal{S} \in s\mathcal{C}(\mathcal{X}, \mathcal{Y})$, $\mathcal{R} \in s\mathcal{C}(\mathcal{Y}, \mathcal{X})$ such that

$$0 \neq \mathcal{RES} \propto \mathcal{I}_{\mathcal{X}}. \quad (7)$$

[2] Koashi, Masato, and Masahito Ueda. "Reversing measurement and probabilistic quantum error correction." Physical review letters 82.12 (1999): 2598.

[3] Fern, Jesse, and John Terilla. "Probabilistic quantum error correction." arXiv preprint quant-ph/0209058 (2002).

[4] Ashikhmin, Alexei E., et al. "Quantum error detection. I. Statement of the problem." IEEE transactions on information theory 46.3 (2000): 778-788.

[5] Wang, Kelvin, Xinyu Zhao, and Ting Yu. "Environment-assisted quantum state restoration via weak measurements." Physical Review A 89.4 (2014): 042320.

Unambiguous quantum error correction

Lemma

Let $\mathcal{E} = \{\mathcal{E}_i\} \in \mathcal{C}(\mathcal{Y},\mathcal{Y})$. The following conditions are equivalent:

A. $\exists\ \mathcal{S} \in s\mathcal{C}(\mathcal{X},\mathcal{Y}),\ \mathcal{R} \in s\mathcal{C}(\mathcal{Y},\mathcal{X}):$

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- **B** $\exists\, 0 \leq R \leq \mathbb{1}_{\mathcal{Y}},\ \mathcal{S} = \{S_k\} \in s\mathcal{C}(\mathcal{X}, \mathcal{Y})$:

$$\{\sqrt{R}\mathcal{E}_i S_k\} = \{A_i\} : \quad A_i \neq 0, \quad A_j^\dagger A_i \propto \delta_{ij} \mathbb{1}_{\mathcal{X}}. \tag{9}$$

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C. $\exists\, 0 \leq R \leq \mathbb{1}_{\mathcal{Y}},\ \mathcal{S} = \{S_k\} \in s\mathcal{C}(\mathcal{X}, \mathcal{Y})$, matrix $\alpha \neq 0$:

$$S_l^\dagger \mathcal{E}_j^\dagger R \mathcal{E}_i S_k = \alpha_{ji,lk} \mathbb{1}_{\mathcal{X}}. \tag{10}$$

uQEC on mixed states

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For some noise channels $\mathcal{E}$, the **non-unitary** encoding strategies $\mathcal{S}$ provide advantage in **unambiguous** quantum error correction.

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Let $\dim(\mathcal{X}) = d$.

Main result

$$p_0 := \frac{1}{d} \max_{\mathcal{S},\mathcal{R}} \left(\mathrm{tr} \left(\mathcal{RES}(\mathbb{1}_{\mathcal{X}}) \right) \,:\, \mathcal{RES} \propto \mathcal{I}_{\mathcal{X}} \right). \tag{11}$$

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Assume that for $\mathcal{S} \in s\mathcal{C}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{R} \in s\mathcal{C}(\mathcal{Y}, \mathcal{X})$ it holds $\mathcal{R}\mathcal{E}\mathcal{S} = p\mathcal{I}_{\mathcal{X}}$.

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$$\operatorname{rank}(J(\mathcal{S})) = 1 \implies p < p_0. \tag{12}$$

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What kind of noises have this property?

Example

$\mathcal{X} = \mathbb{C}^2$, $\mathcal{Y} = \mathbb{C}^4$
$V_1 = |0\rangle \otimes \mathbb{1}_2$
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$D = \mathrm{Diag}^\dagger(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad \lambda_i < 1, \ \lambda_1 \neq \lambda_4$ or $\lambda_2 \neq \lambda_3$

We will take $(\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}, \lambda_4 = \frac{1}{4})$

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$$\mathcal{E}(Y) := |0\rangle\langle 0| \otimes \left(V_1^\dagger \sqrt{R} Y \sqrt{R} V_1 + V_2^\dagger \sqrt{R} Y \sqrt{R} V_2 \right) + |1\rangle\langle 1| \otimes \mathrm{tr}\left([\mathbb{1}_4 - R] Y \right) \rho_* \tag{13}$$

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Optimal $\mathcal{R}$?

$$\mathcal{R}(A \otimes B) = \mathrm{tr}\left(A |0\rangle\langle 0|\right) B \tag{14}$$

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Simplified task

$$\max\left(\sum_k \|\psi_k\|^2 : \quad \sum_k (\langle\psi_k| \otimes \mathbb{1}_2) R^{-1} (|\psi_k\rangle \otimes \mathbb{1}_2) \le \mathbb{1}_2\right) \tag{17}$$

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Example

$$V_1 = |0\rangle \otimes \mathbb{1}_2$$
$$V_2 = |1\rangle \otimes \mathbb{1}_2$$
$$D = \mathrm{Diag}^\dagger\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right)$$
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Optimal strategy $\mathcal{S} = \{\sqrt{R}^{-1} A_k\}$:

$$A_1 = \tfrac{1}{\sqrt{5}}|0\rangle \otimes \mathbb{1}_2$$
$$A_2 = \tfrac{1}{\sqrt{5}}|1\rangle \otimes \mathbb{1}_2$$

Example summary

$$
\begin{aligned}
&|00\rangle\langle 00| \otimes \rho \\
\rightarrow & U_{\mathcal{S}} \left(|00\rangle\langle 00| \otimes \rho \right) U_{\mathcal{S}}^{\dagger} \\
\rightarrow & \mathrm{tr}_1 \left(U_{\mathcal{S}} \left(|00\rangle\langle 00| \otimes \rho \right) U_{\mathcal{S}}^{\dagger} \right) \\
\rightarrow & \mathcal{E} \left(\mathrm{tr}_1 \left(U_{\mathcal{S}} \left(|00\rangle\langle 00| \otimes \rho \right) U_{\mathcal{S}}^{\dagger} \right) \right) \\
\rightarrow & (\langle 0| \otimes \mathbb{1}_2) \mathcal{E} \left(\mathrm{tr}_1 \left(U_{\mathcal{S}} \left(|00\rangle\langle 00| \otimes \rho \right) U_{\mathcal{S}}^{\dagger} \right) \right) (|0\rangle \otimes \mathbb{1}_2) \\
\rightarrow & 0.4\rho
\end{aligned}
$$

Thank you for your attention!