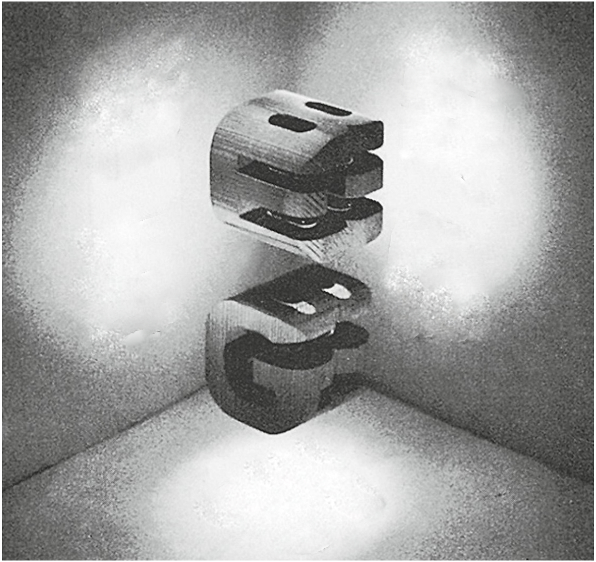
How separability decides marginal problems [arXiv:2008.02124]

X.-D. Yu, T. Simnacher, N. Wyderka, H. C. Nguyen, O. Gühne

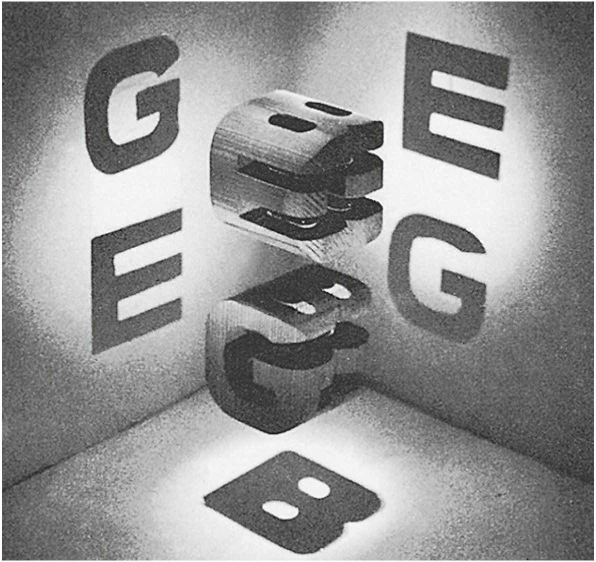
Universität Siegen

Marginal Problems

 [1]

[1] D. R. Hofstadter, Gödel, Escher, Bach: ein Endloses Geflochtenes Band (1992)

Marginal Problems



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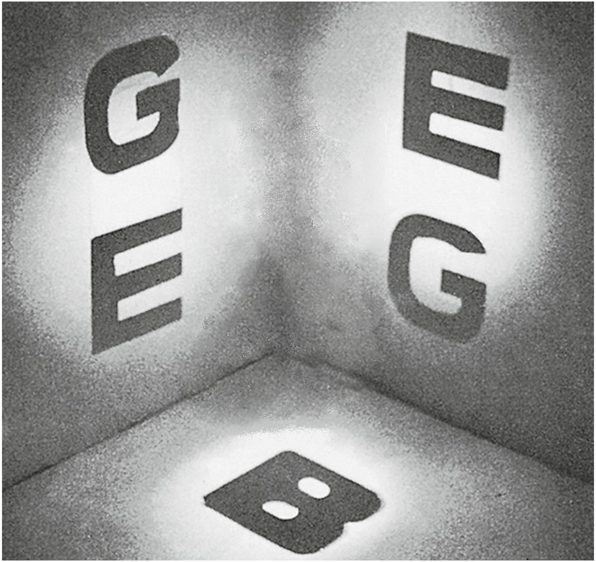

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Marginal Problems

							1	
4								
	2							
				5		4		7
		8				3		
		1		9				
3			4			2		
	5		1					
			8		6			

[2] [3]

[2] A. M. Herzberg, M. R. Murty, Notices of the AMS, **54**, 708 (2007).

[3] G. McGuire, B. Tugemann, G. Civario, arxiv:1201.0749.

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6	9	3	7	8	4	5	**1**	2
4	8	7	5	1	2	9	3	6
1	**2**	5	9	6	3	8	7	4
9	3	2	6	**5**	1	**4**	8	**7**
5	6	**8**	2	4	7	**3**	9	1
7	4	**1**	3	**9**	8	6	2	5
3	1	9	**4**	7	5	**2**	6	8
8	**5**	6	**1**	2	9	7	4	3
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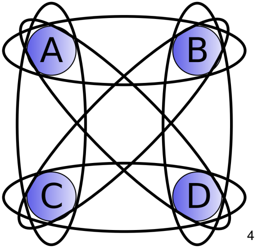

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[4] N. Wyderka, F. Huber, O. Gühne, Phys. Rev. A **97**, 060101 (2018)

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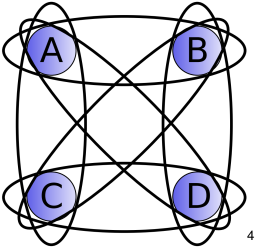

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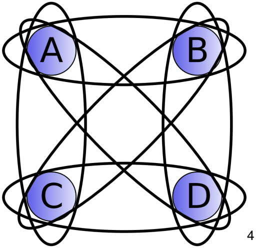
- Example: If we have a bipartite state ρ_{AB} with given marginals $\rho_A = \operatorname{Tr}_B \rho_{AB} = \frac{\mathbb{1}}{d}$ and $\rho_B = \operatorname{Tr}_A \rho_{AB} = \frac{\mathbb{1}}{d}$, then...



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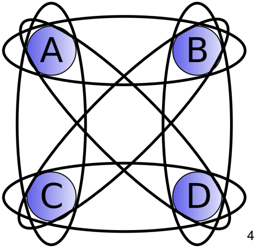

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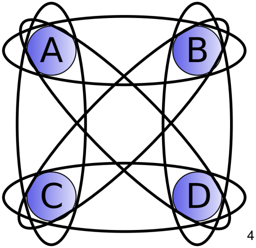

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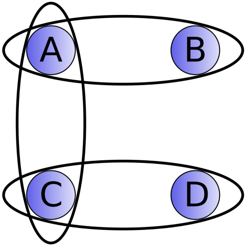

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Motivation

- What do parts of quantum states tell about the whole?[5] What does it tell us about entanglement?

[5]N. Wyderka, Dissertation (2020)
[6]F. Huber, O. Gühne, Phys. Rev. Lett. **117**, 010403 (2016)
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- Apart from the *existence* question, it is also interesting to ask whether the global state is uniquely determined by the marginals. This decides the uniqueness of the ground state of a local Hamiltonian[6] $H = A_1 \otimes A_2 \otimes \mathbb{1} + B_1 \otimes \mathbb{1} \otimes B_3 + \mathbb{1} \otimes C_2 \otimes C_3$.

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- One can consider further restrictions, e.g., that the global state has to be fermionic which is important in quantum chemistry[7].

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Semi-definite programming[8]

- primal problem:

$$\min_{X,\boldsymbol{x}} \quad \sum_{j=1}^{n} x_j c_j$$
$$\text{s.t.} \quad \sum_{j=1}^{n} x_j F_j - F_0 = X \geq 0.$$

[8] L. Vandenberghe and S. Boyd, SIAM Review **38**, 49 (1996).

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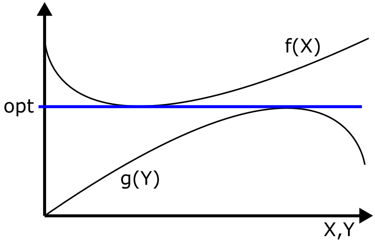

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$$\max_{Y} \quad \mathrm{Tr}(Y F_0)$$
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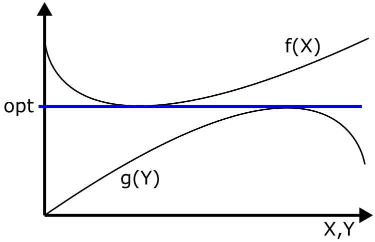

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- Example:

$$\text{find} \quad \rho_{AB}$$

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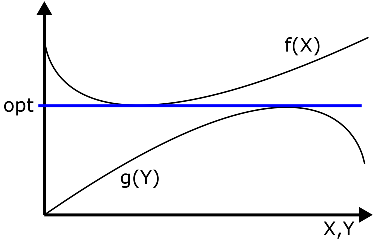

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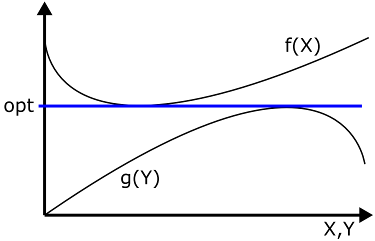

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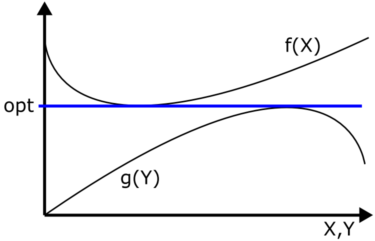

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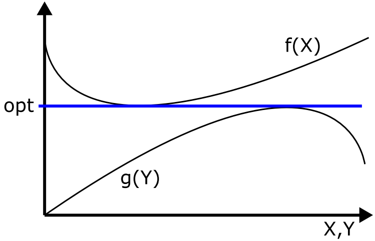

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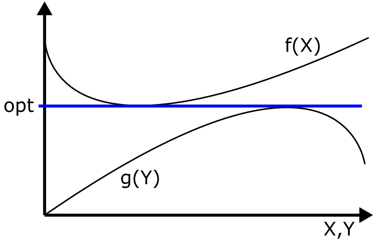

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- SDPs are *easy* and *reliable* to solve with a computer.
- Analytical bounds and sometimes proofs can be obtained.

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Pure quantum marginal problem

- The quantum marginal problem becomes way more complicated if we require the global state to be pure:

$$\begin{aligned} \text{find} \quad & |\varphi\rangle \\ \text{s.t.} \quad & \mathrm{Tr}_{I^C}(|\varphi\rangle\langle\varphi|) = \rho_I,\ I \in \mathcal{I}, \end{aligned}$$

where I^C is the complement of I, i.e., all particles which are not in I and $\mathcal{I}$ are some subsets of particles.

[9] A. Klyachko, arXiv:quant-ph/0409113.

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- So far, there is only a structured approach when the marginals are non-overlapping[9], i.e., $I_1 \cap I_2 = \emptyset$ for $I_1, I_2 \in \mathcal{I}$.

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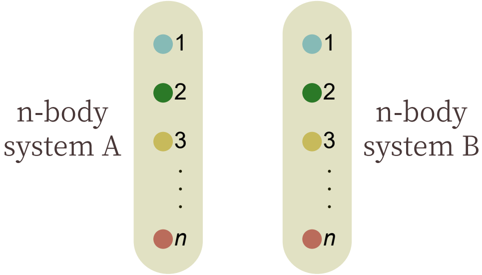

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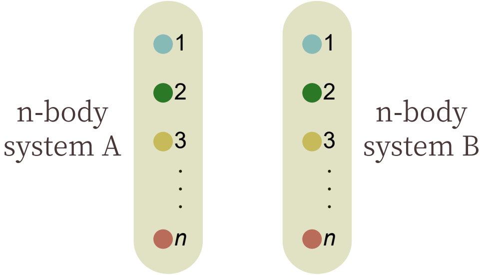

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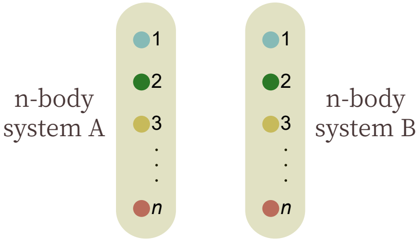

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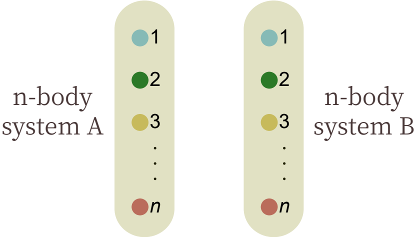

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- In this case, we have

$$\begin{aligned} \mathrm{Tr}_{A_{I^c}}(\Phi_{AB}) &= (\mathrm{Tr}_{I^c}|\varphi\rangle\langle\varphi|) \otimes |\varphi\rangle\langle\varphi| \\ &= \rho_I \otimes |\varphi\rangle\langle\varphi| \\ &= \rho_I \otimes \mathrm{Tr}_A(\Phi_{AB}). \end{aligned}$$

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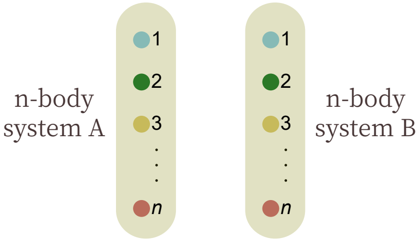

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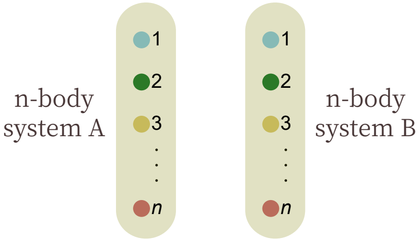

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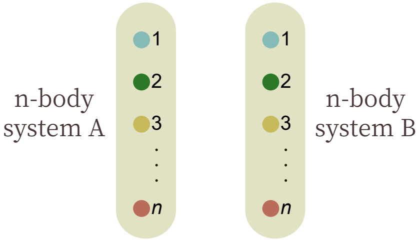

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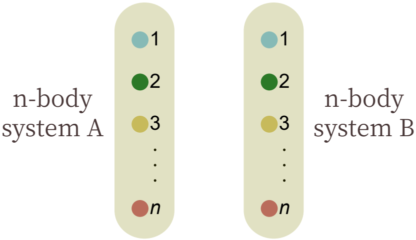

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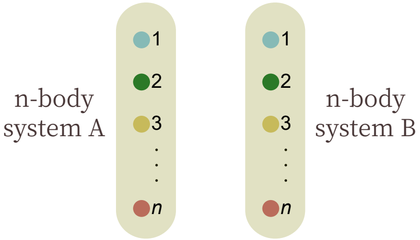

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- Let X be any Hermitian matrix such that $\mathrm{Tr}(X\rho_I) = 0$. Then,

$$\begin{aligned} 0 &= \mathrm{Tr}[(X \otimes X)(\rho_I \otimes \rho_I)] \\ &= \sum_\mu p_\mu \,\mathrm{Tr}[(X \otimes X)(\sigma_\mu \otimes \sigma_\mu)] \\ &= \sum_\mu p_\mu [\mathrm{Tr}(X\sigma_\mu)]^2. \end{aligned}$$

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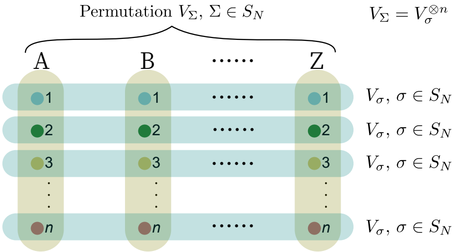
- Thus, together with normalization $\sigma_\mu = \rho_I$ and hence, all the $|\psi_\mu\rangle$ are solutions to the marginal problem.

Quantum de Finetti theorem

- For a separable, symmetric state $\Phi_{AB} = \sum_\mu p_\mu |\psi_\mu\rangle\langle\psi_\mu| \otimes |\psi_\mu\rangle\langle\psi_\mu|$, it is easy to write down a symmetric extension $\Phi_{ABC\ldots Z} = \sum_\mu p_\mu |\psi_\mu\rangle\langle\psi_\mu| \otimes \cdots \otimes |\psi_\mu\rangle\langle\psi_\mu|$.



[10] C. M. Caves, C. A. Fuchs, R. Schack, J. Math. Phys. **43**, 4537 (2002).

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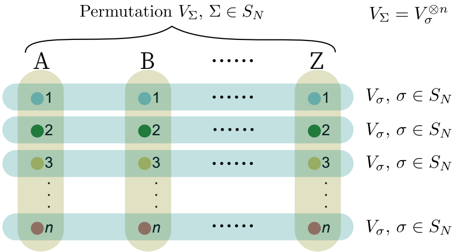
- The quantum de Finetti theorem [10][11] tells us that also the reverse is true: If there exist extensions $\Phi_{ABC\ldots Z}$ on the symmetric subspace for any number of extra particles such that $\Phi_{AB} = \mathrm{Tr}_{C\ldots Z}\, \Phi_{ABC\ldots Z}$, then $\Phi_{AB} = \sum_\mu p_\mu |\psi_\mu\rangle\langle\psi_\mu| \otimes |\psi_\mu\rangle\langle\psi_\mu|$.



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Main results

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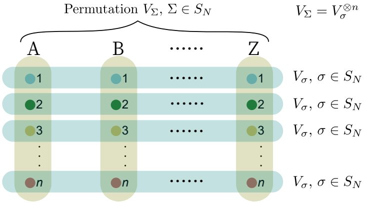

Due to the quantum de Finetti theorem, these are also equivalent and can be tested with the following hierarchy for $N \geq 2$:

$$\begin{aligned} \text{find} \quad & \Phi_{AB\cdots Z} \\ \text{s.t.} \quad & P_N^+ \Phi_{AB\cdots Z} P_N^+ = \Phi_{AB\cdots Z}, \\ & \Phi_{AB\cdots Z} \geq 0,\ \ \mathrm{Tr}(\Phi_{AB\cdots Z}) = 1, \\ & \mathrm{Tr}_{A_{I^c}}(\Phi_{AB\cdots Z}) = \rho_I \otimes \mathrm{Tr}_A(\Phi_{AB\cdots Z}) \quad \forall\, I \in \mathcal{I}. \end{aligned}$$

Application: **A**bsolutely **M**aximally **E**ntangled states

- An AME(n, d) state with n particles and local dimension d is a pure state with all $\lfloor \frac{n}{2} \rfloor$-marginals maximally mixed.

[12] https://www.tp.nt.uni-siegen.de/+fhuber/ame.html

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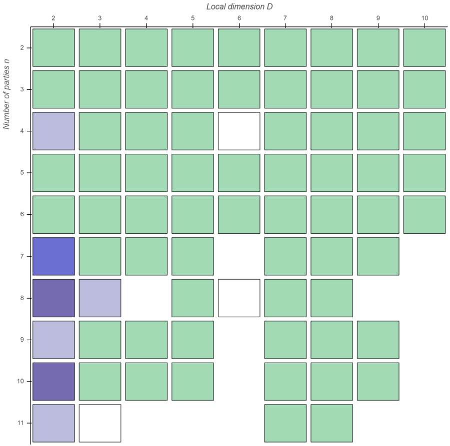

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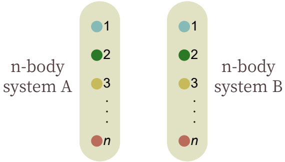

- Similarly, we can consider

$$\Phi_{AB} = \frac{1}{\mathcal{N}} \sum_{\pi} V_{\pi}^{\otimes 2} \left[\int dU_1 \cdots dU_n\, U_1^{\otimes 2} \otimes \cdots \otimes U_n^{\otimes 2} |\varphi\rangle\langle\varphi| \otimes |\varphi\rangle\langle\varphi| \right.$$
$$\left. (U_1^{\dagger})^{\otimes 2} \otimes \cdots \otimes (U_n^{\dagger})^{\otimes 2} \right] (V_{\pi}^{\dagger})^{\otimes 2}$$

as a solution to the problem.

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- This solution satisfies the symmetries
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  We can add them to the problem.

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  We can add them to the problem.
- Then, Schur-Weyl duality tells us that we can write

$$\Phi_{AB} = \sum_{i=0}^{n} x_i \mathcal{P}\{V^{\otimes i} \otimes \mathbb{1}^{\otimes (n-i)}\}.$$

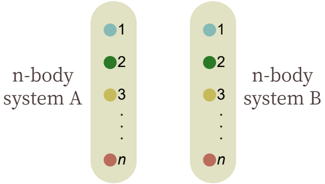
Application: **A**bsolutely **M**aximally **E**ntangled states

Using symmetries, we show that the existence of an AME(n, d) state is equivalent to the separability of

$$\Phi_{AB} = \sum_{i=0}^{n} x_i \mathcal{P}\{V^{\otimes i} \otimes \mathbb{1}^{\otimes (n-i)}\},$$

where

$$x_i = \frac{(-1)^i}{(d^2-1)^n} \sum_{l=0}^{n} \sum_{k=0}^{l} \frac{(-1)^l \binom{i}{k} \binom{n-i}{l-k}}{\min\{d^{i+2l-2k}, d^{n+i-2k}\}}.$$



Take-home message

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- Watch out for new results on the arXiv in the upcoming weeks!