Universal witnesses of vanishing energy gap

Konrad Szymański

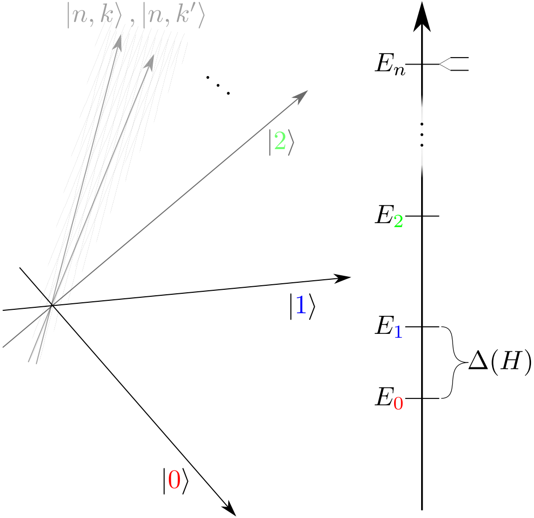
konrad.jan.szymanski@gmail.com

Uniwersytet Jagielloński
June 8, 2020

Plan

- What is spectral gap and what is it useful for?
- Geometric approach: numerical range.
- Detection in XX model.

Spectral gap $\Delta(H)$ is the difference of energies of two lowest eigenstates of H



H – arbitrary Hamiltonian.

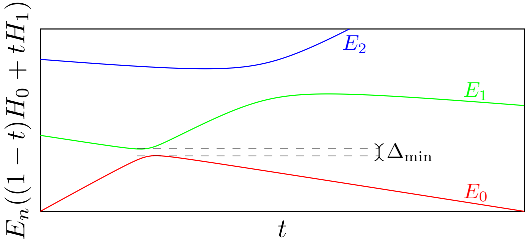
$$H|n, k\rangle = E_n |n, k\rangle$$

Spectral gap is $\Delta(H) = E_1 - E_0$, excluding degeneracy.

Spectral gap $\Delta(H)$ has critical importance in some fields

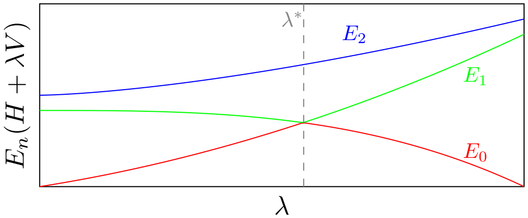
Speed limit of adiabatic quantum computers

To obtain $|0 : H_1\rangle$, start with $|0 : H_0\rangle$, evolve with $H(t) = (1 - \frac{t}{T})H_0 + \frac{t}{T}H_1$. Total time $T \propto \Delta_{\min}^{-3}$.



Indicator of quantum phase transitions

- $\Delta(H) \to 0$ – change of ground state structure

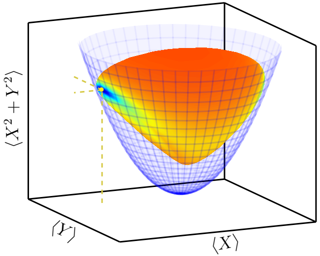


Numerical ranges help in understanding QM

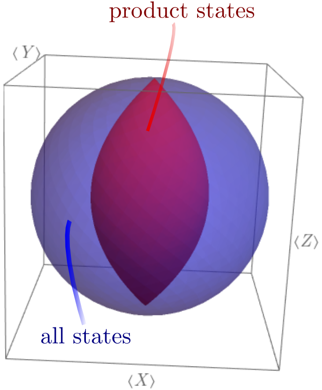
$$W(H_1, \dots, H_k) = \{(\langle\psi| H_1 |\psi\rangle, \dots, \langle\psi| H_k |\psi\rangle) : \||\psi\rangle\| = 1\}$$

Range of possible expectation values of observables or Hamiltonians.
(sometimes with density operators instead of pure states)

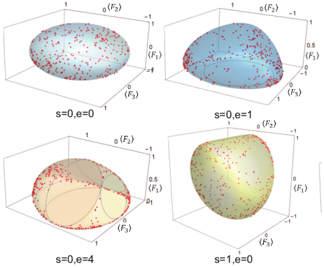
Uncertainty relations



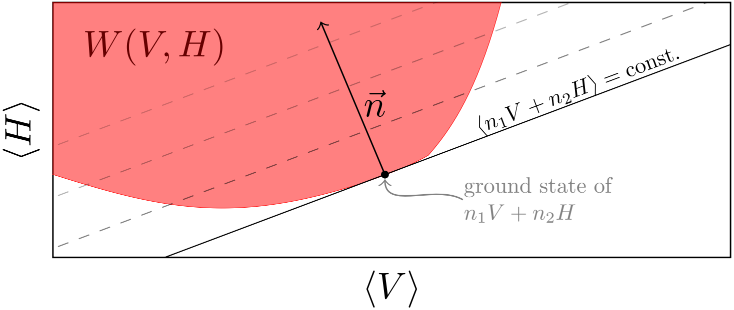
Entanglement structure



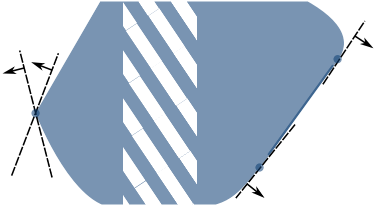
Phase transitions



Features of $\partial W(V, H)$ determine properties of H, V



Entire dynamics of $H + \lambda V$ is encoded in the boundary $\partial W(V, H)$.



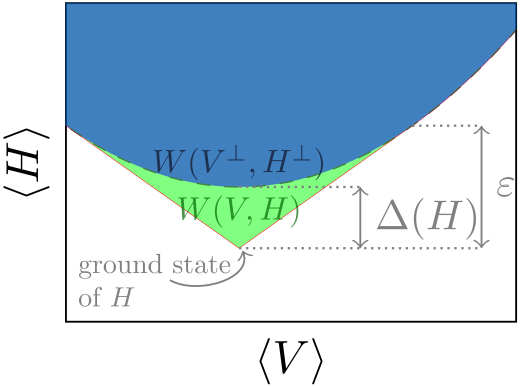
Cusp

ground state for many $\vec{n}$
$\implies$ eigenstate of H and V.

Segment

discontinous change of ground state of $H + \lambda V$.

A certain kind of cusp in $\partial W(V,H)$ proves $\Delta(H)=0$



a) Cusp $\implies$ ground state of H is an eigenstate of V.

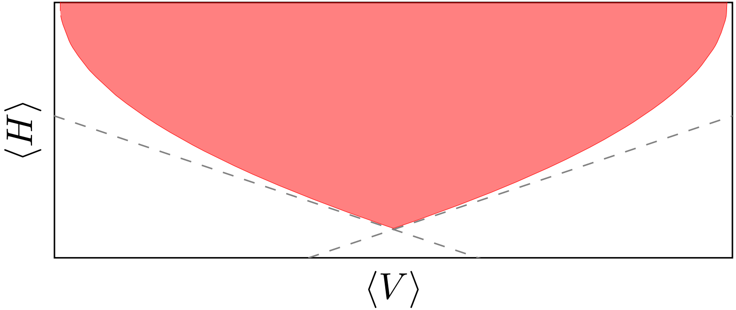
b) $H = H_{(0)} \oplus H^\perp$, $V = V_{(0)} \oplus V^\perp$.

c) $W(V,H) = \operatorname{conv} W(V^\perp, H^\perp) \cup \text{cusp}$

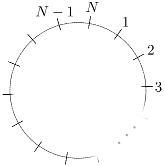
d) Convex hull of separated sets $\implies$ segments of height $\varepsilon \geq \Delta(H)$.

e) $\epsilon = 0 \implies$ sets not separated, $E_1 \to E_0$.

Anticipated behavior is visible in 1-D XX model



Cyclic boundary conditions: $\sigma_{N+1} = \sigma_1$.

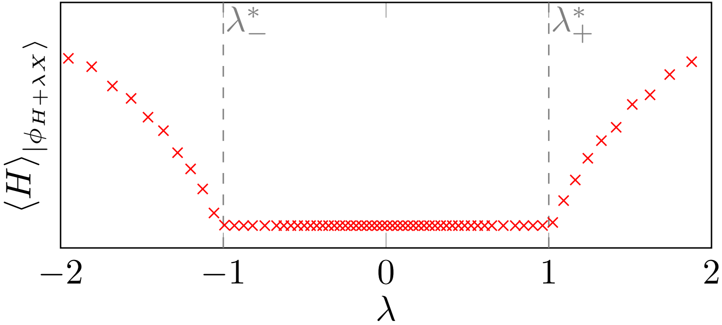


$$H = \sum_{n=1}^{N} \sigma_n^{(x)} \sigma_{n+1}^{(x)} + \sigma_n^{(y)} \sigma_{n+1}^{(y)},$$

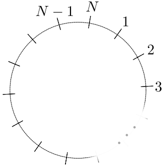
$$V = \sum_{n=1}^{N} \sigma_{n-1}^{(x)} \sigma_n^{(z)} \sigma_{n+1}^{(y)} - \sigma_{n-1}^{(y)} \sigma_n^{(z)} \sigma_{n+1}^{(x)}.$$

Numerical results (MPS): $N \approx 150$.

Anticipated behavior is visible in 1-D XX model



Cyclic boundary conditions: $\sigma_{N+1} = \sigma_1$.

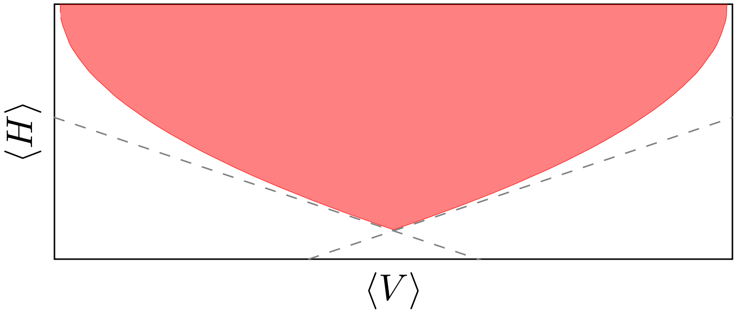


$$H = \sum_{n=1}^{N} \sigma_n^{(x)} \sigma_{n+1}^{(x)} + \sigma_n^{(y)} \sigma_{n+1}^{(y)},$$

$$V = \sum_{n=1}^{N} \sigma_{n-1}^{(x)} \sigma_n^{(z)} \sigma_{n+1}^{(y)} - \sigma_{n-1}^{(y)} \sigma_n^{(z)} \sigma_{n+1}^{(x)}.$$

Numerical results (MPS): $N \approx 150$.

Main result: It is possible to verify $\Delta(H) = 0$ in *XX* model.



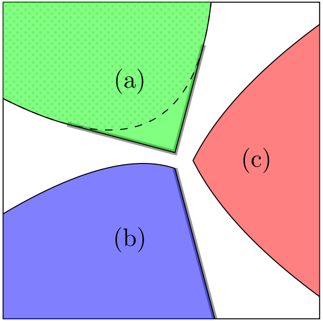
The cusp has no connecting segments $\implies \Delta(H) = 0$!

$$H = \sum_{n=1}^{N} \sigma_n^{(x)} \sigma_{n+1}^{(x)} + \sigma_n^{(y)} \sigma_{n+1}^{(y)},$$

$$V = \sum_{n=1}^{N} \sigma_{n-1}^{(x)} \sigma_n^{(z)} \sigma_{n+1}^{(y)} - \sigma_{n-1}^{(y)} \sigma_n^{(z)} \sigma_{n+1}^{(x)}.$$

Conclusions

- Geometry of quantum states aids in undestanding of phase transitions and related phenomena.
- In particular, it is possible to prove that a certain hamiltonian H has vashining energy gap $\Delta(H) = 0$.
- "Geometric proof" is not a completely theoretical construct: it works for XX model!



Bibliography

1. Wawel castle painting by Ulf Andersson/ArtMagenta
2. Tameem Albash and Daniel A. Lidar. *"Adiabatic quantum computation."* Reviews of Modern Physics 90.1 (2018): 015002.
3. Jie Xie et al. *"Observing geometry of quantum states in a three-level system."* arXiv preprint arXiv:1909.05463 (2019).
4. Konrad Szymański and Karol Życzkowski. *"Geometric and algebraic origins of additive uncertainty relations."* Journal of Physics A: Mathematical and Theoretical 53.1 (2019): 015302.