

# Entanglement witnesses: overview of the technique and a new construction

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# Overview

Introduction

Realignment criterion and beyond

Our result - linear witnesses from non-linear criterion



## Separable states and Schmidt decomposition

**Definition:** A pure state  $\rho \in \mathcal{B}(\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_n})$  is called *separable* if it is represented by a product state vector:

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It generalises to decomposition of a bipartite state:

$$\rho = \sum_{i=1}^{\min(d_1^2, d_2^2)} \lambda_i F_i \otimes G_i, \quad (3)$$

where  $\langle F_i | F_j \rangle_{HS} = \delta_{ij}$ ,  $\langle G_i | G_j \rangle_{HS} = \delta_{ij}$ ,  $\sum_i \lambda_i^2 = 1$  and  $F_i$ 's and  $G_i$ 's are hermitian.

(not a separable decomposition -  $F_i$  and  $G_i$  in general not positive!)



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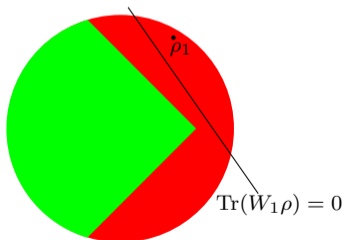


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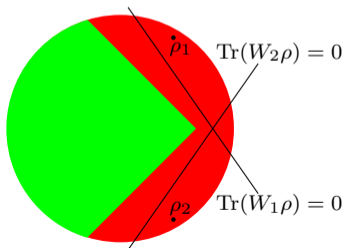


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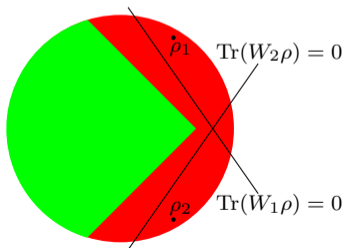
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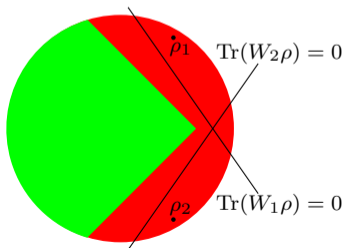


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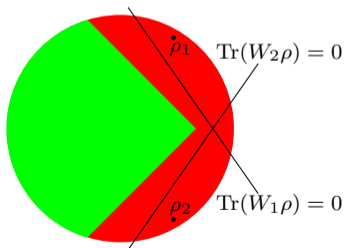


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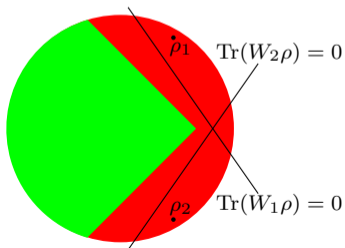


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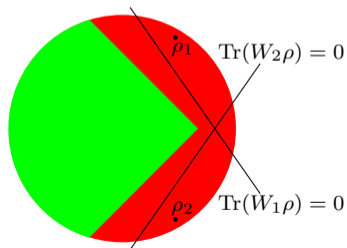


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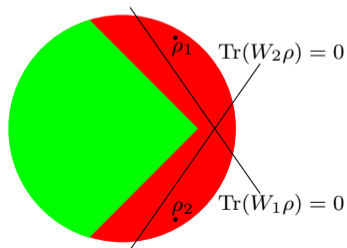


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Positivity of  $(I_{d_1} \otimes \Phi)(\rho)$  is equivalent to positive expected value of a family of entanglement witnesses:  $\{A \otimes B W_\Phi A^\dagger \otimes B^\dagger\}$ .



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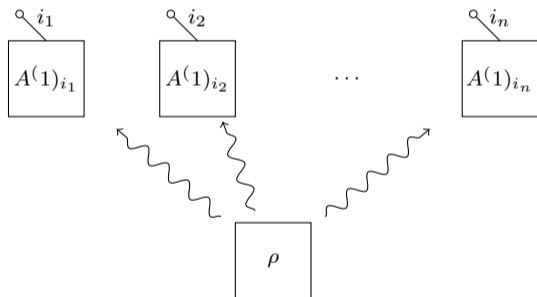
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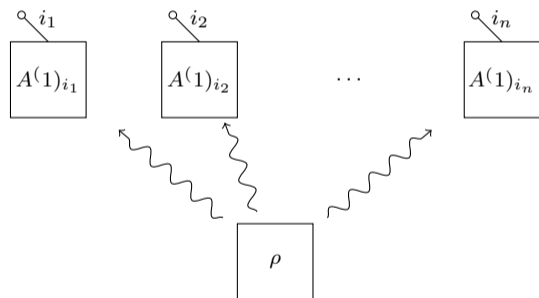




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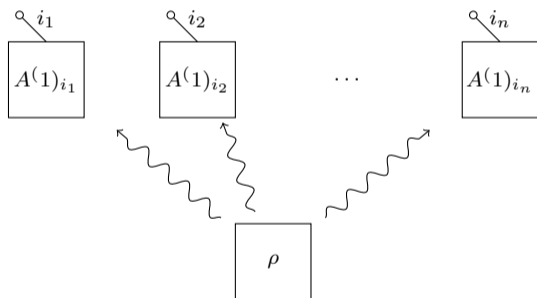
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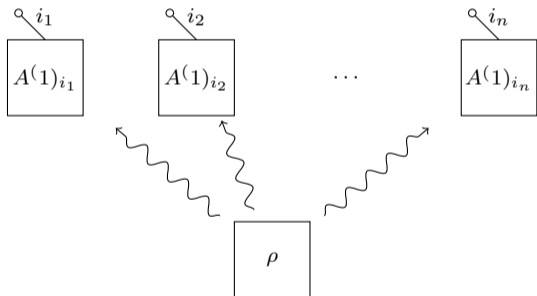
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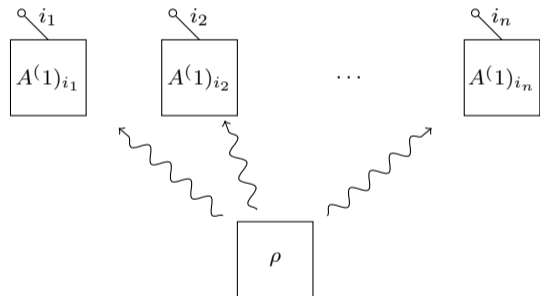
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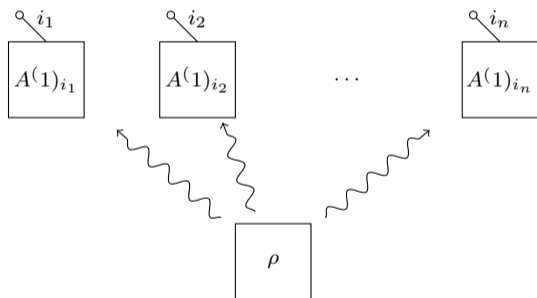
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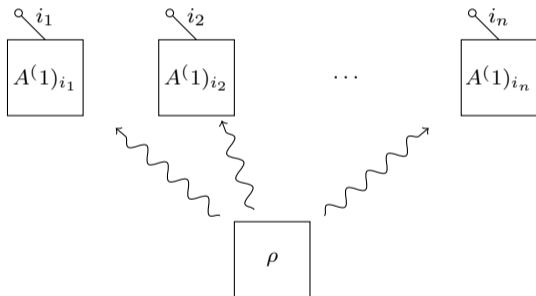
In higher dimensions exists PPT entangled states.



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For systems  $2 \times 2$  and  $2 \times 3$  also  $\Leftarrow$ .

In higher dimensions exists PPT entangled states.

Other criteria or other maps are necessary to detect such entanglement.



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and the realignment criterion is equivalent to family of witnesses:

$$W_O = I - \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij}, \quad (8)$$

parametrised by isometry matrices.



## Enhanced realignment criterion and other C-based criteria

These witnesses can be strengthened by a non-linear correction:

$$\widetilde{W}_O = I - \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij} - \frac{1}{2}(\text{Tr}\rho_A^2 + \text{Tr}\rho_B^2). \quad (9)$$



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- **de Vicente criterion:** Let  $\widetilde{C}$  be  $C$  with first row and first column removed (restrict to traceless subspaces). Then

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$$\|\hat{C}\|_1 \leq \sqrt{\frac{d_1 + 1}{2d_1}} \sqrt{\frac{d_2 + 1}{2d_2}} \quad (12)$$



## Generalisation of the linear criteria arXiv:2001.08258

Redefine:  $C_{x,y} = \text{diag}\{x, 1, \dots, 1\}C\text{diag}\{y, 1, \dots, 1\}$ .

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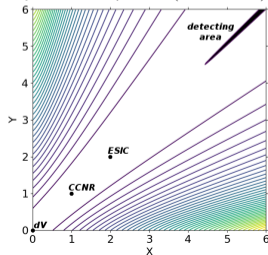
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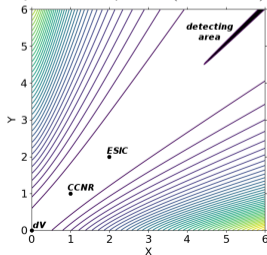
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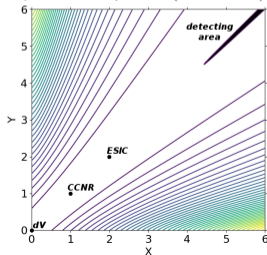
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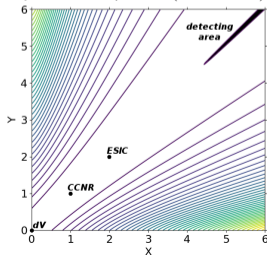
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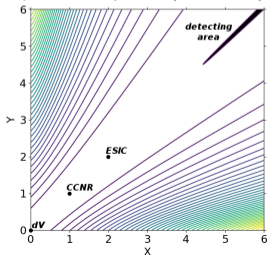
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We have proven the following: for  $\rho$  - separable:

$$\forall x_1, \dots, x_n \quad \|C_{x_1, \dots, x_n}\|_1 \leq \prod_i \sqrt{\frac{d_i - 1 + x_i^2}{d_i}} \quad (16)$$



# Limit of bipartite case arXiv:2002.00646

We find a family of witnesses corresponding to our criterion:

$$W_{O,x,y} = a_{xy} G_0^A \otimes G_0^B + x G_0^A \otimes \left( \sum_{\beta>0} O^{0\beta} G_\beta^B \right) + y \left( \sum_{\beta>0} O^{\alpha 0} G_\alpha^A \right) \otimes G_0^B + \sum_{\alpha,\beta>0} O^{\alpha\beta} G_\alpha^A \otimes G_\beta^B, \quad (17)$$

where  $a_{xy} = \left( \sqrt{d_A - 1 + x^2} \sqrt{d_B - 1 + y^2} + xy O^{00} \right)$ .  $\lim_{x,y \rightarrow \infty} O^{00} = -1$ , otherwise  $\lim W_{O,x,y} \sim I \otimes I$ .

We take:

$$O = \left[ \begin{array}{c|c} -\sqrt{1 - \eta^2/r^2} & \eta/r \mathbf{v}^T \\ \hline \eta/r \mathbf{u} & \mathbf{0} \end{array} \right] \quad (18)$$

(up to  $O(r^2)$ ), where  $\mathbf{u}$  and  $\mathbf{v}$  are unit vectors satisfying  $\mathbf{u} = \mathbf{Ov} / \sqrt{1 - \eta^2/r^2} \xrightarrow{r \rightarrow \infty} \mathbf{Ov}$  and get the limit:

$$\begin{aligned} W^\infty = & \frac{(d_B - 1) \cot \theta + (d_A - 1) \tan \theta + \eta^2 \sin \theta \cos \theta}{2} \frac{I_{d_A}}{\sqrt{d_A}} \otimes \frac{I_{d_B}}{\sqrt{d_B}} + \eta \cos \theta \frac{I_A}{\sqrt{d_A}} \otimes \sum_{\beta>0} v^\beta G_\beta^B \\ & + \eta \sin \theta \sum_{\alpha>0} (\tilde{O}v)^\alpha G_\alpha^A \otimes \frac{I_B}{\sqrt{d_B}} + \sum_{\alpha,\beta>0} \tilde{O}^{\alpha\beta} G_\alpha^A \otimes G_\beta^B \end{aligned} \quad (19)$$



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First we prove, that (the simple part):

$$\|C(\rho - \rho_A \otimes \rho_B)\|_1 \leq \sqrt{1 - \text{Tr}\rho_A^2} \sqrt{1 - \text{Tr}\rho_B^2} \Rightarrow \|C_{xy}(\rho)\|_1 \leq \sqrt{\frac{d_A - 1 + x^2}{d_A}} \sqrt{\frac{d_B - 1 + y^2}{d_B}} \quad (20)$$

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for all  $x, y$ . Hence no correlation tensor based criterion can detect more than the enhanced realignment criterion. Now we consider the limit witnesses:

$$\begin{aligned} W^\infty = & \frac{(d_B - 1) \cot \theta + (d_A - 1) \tan \theta + \eta^2 \sin \theta \cos \theta}{2} \frac{I_{d_A}}{\sqrt{d_A}} \otimes \frac{I_{d_B}}{\sqrt{d_B}} + \eta \cos \theta \frac{I_A}{\sqrt{d_A}} \otimes \sum_{\beta > 0} v^\beta G_\beta^B \\ & + \eta \sin \theta \sum_{\alpha > 0} (\tilde{O}v)^\alpha G_\alpha^A \otimes \frac{I_B}{\sqrt{d_B}} + \sum_{\alpha, \beta > 0} \tilde{O}^{\alpha\beta} G_\alpha^A \otimes G_\beta^B \end{aligned} \quad (21)$$

and look for the minimum of their expected values for a given state  $\rho$ .



# Equivalence of criteria arXiv:2002.00646

First we prove, that (the simple part):

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- $\mathbf{O} = UV^T$ , where  $UDV^T$  is a SVD of  $\rho - \rho_A \otimes \rho_B$
- $\eta = \frac{\sqrt{d_A d_B}}{\sin \theta \cos \theta} \left\| \frac{\cos \theta}{\sqrt{d_A}} \tilde{\rho}_B + \frac{\sin \theta}{\sqrt{d_B}} \tilde{O}^T \tilde{\rho}_A \right\|$
- $v = - \frac{\frac{A \cos \theta}{\sqrt{d_A}} \tilde{\rho}_B + \frac{A \sin \theta}{\sqrt{d_B}} \tilde{O}^T \tilde{\rho}_A}{\left\| \frac{A \cos \theta}{\sqrt{d_A}} \tilde{\rho}_B + \frac{A \sin \theta}{\sqrt{d_B}} \tilde{O}^T \tilde{\rho}_A \right\|}$



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Hence if the enhanced realignment criterion detects entanglement in  $\rho$ , then it is detected by a witness of a form  $W^\infty$  as well, hence it is also detected by  $W_{O,x,y}$  for large enough  $x, y$ .





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Thank you for your attention!



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