Entropic uncertainty relations

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13 I 2020

Plan

1. Canonical Robertson uncertainty relations and problems with them
2. Idea of entropic uncertainty
3. Finite dimensional spaces at special case
4. Maassen and Uffniks general model
5. Generalizations and problems with them.

Conventions

- Hilbert space $\mathscr{H}$, with the sesquilinear, Hermitian scalar product $\langle\cdot,\cdot\rangle$

$$\langle a\psi, \phi\rangle = a\langle\psi, \phi\rangle$$

  Norm

$$\|\psi\|^2 := \langle\psi, \psi\rangle > 0$$

  for any $\psi \neq 0$.

$$\langle A\rangle_\psi = \langle A\psi, \psi\rangle$$

- By default $\|\psi\| = 1$ and $A^* = A$ for any vector and operator.

Canonical Robertson uncertainty

- $$\left\|(xA+iB)\psi\right\|^2 = \langle A\rangle_\psi^2 x^2 + \langle i[A,B]\rangle_\psi x + \langle B\rangle_\psi^2 \geqslant 0$$

  This gives
  $$\langle i[A,B]\rangle_\psi^2 - 4\langle A\rangle_\psi^2 \langle B\rangle_\psi^2 \leqslant 0$$

- $$[A-\langle A\rangle \mathbb{I}, B-\langle B\rangle \mathbb{I}] = [A,B]$$

  as everything commutes with $\mathbb{I}$.
- We obtain the Robertson relations:
  $$\Delta_\psi^2 A \cdot \Delta_\psi^2 B \geqslant \frac{1}{4}\langle i[A,B]\rangle_\psi^2$$

  where
  $$\Delta_\psi^2 A := \left\|(A-\langle A\rangle \mathbb{I})\psi\right\|^2$$

Problems

$$\Delta^2_\psi A \cdot \Delta^2_\psi B \geqslant \tfrac{1}{4} \langle i[A,B] \rangle^2_\psi$$

- In particular, this yields

  $$\Delta^2_\psi X \cdot \Delta^2_\psi P \geqslant \tfrac{1}{4}$$

  as $[X, P] = i\mathbb{I}$.
- General Robertson relations above should not be considered as real uncertainties as the bound itself depends on the state we're talking about. In general, $\langle i[A,B] \rangle_\psi$ could even vanish for some ψ. Taking $A\psi = a\psi$, we easily get

  $$\Delta^2_\psi A = \langle [A,B] \rangle_\psi = 0$$

A puzzle for students:

Why $\langle [X, P] \rangle$ vanishes for no state in $L^2[0;1]$ of potential well?

Deutsches result

- We seek for a relation of the form

$$U(A, B, \psi) \geqslant \mathscr{B}(A, B).$$

- For $\dim \mathscr{H} < \aleph_0$ we can postulate (Deutsch 1983)

$$U(A, B, \psi) = H(A) + H(B),$$

where the **Shannon entropy** is defined as

$$H := -\sum_k p_k \log p_k$$

for arbitrary probability distribution $p : \{1, ..., N\} \to [0; 1]$ (with $\sum_k p_k = 1$). (Note symmetry)

Deutsches result

- In Quantum Mechanics

$$p_k = |\langle \psi, a_k \rangle|^2$$

yields

$$H(A) := -2 \sum_k |\langle \psi, a_k \rangle|^2 \, \log |\langle \psi, a_k \rangle|$$

where $\{a_k\}$ is an orthonormal eigenbasis of A.

- $H(A)/\log 2$ is the deficiency in the information.
- The bound is defined as

$$\mathscr{B}(A, B) := \inf\{ H(A) + H(B) \mid \psi \in \mathscr{H}, \|\psi\| = 1 \}$$

Deutsches result

- $$H(A)+H(B) = -2\sum_{k,l} |\langle \psi, a_k\rangle|^2 |\langle \psi, b_l\rangle|^2 \Big(\log|\langle \psi, a_k\rangle| + \log|\langle \psi, b_l\rangle|\Big)$$

  Expression in the bracket has its minimum for

  $$\psi_{0kl} := \frac{1}{\sqrt{2(1+|\langle a_k, b_l\rangle|)}}\Big(a_k + e^{-i\arg\langle a_k, b_l\rangle} b_l\Big)$$

  (Not a minimum of U!!!), hence

- $$H(A) + H(B) \geqslant \mathscr{B}(A, B) \geqslant 2\log\frac{2}{1+C}$$

  $$C = \max\Big\{ |\langle a_k, b_l\rangle| \ \Big| \ k, l \Big\}$$

Maassen & Uffnik

Maassen & Uffnik (1988)([1]):

We consider a family of maps M such that

$$M_r(p) := \left(\sum_k {p_k}^{r+1}\right)^{1/r}$$

for $r > -1$ except $r = 0$.
On top of that,

$$M_{-1}(p) = \frac{1}{N'}$$

$$M_0(p) = \mathrm{e}^{-H(p)}$$

$$M_\infty(p) = \max\{p_k\}$$

where $N' = \left|\{p_k \neq 0\}\right|$.

[1] Beautifull paper!

Maassen & Uffnik

- Those satisfy (Hardy, Littlewood, Polya 1952)
  - $M_r(p_\pi) = M_r(p)$ for any permutation π
  - $M_r(ap + (1-a)q) \leqslant aM_r(p) + (1-a)M_r(q)$
  - $M_r(p \otimes q) = M_r(p)M_r(q)$
  - $M_r(p)$ continuous, non-decreasing function of r

  for any two finite probability distributions p, q and number $a \in [0;1]$.
- Hence $M_r(p)$ are expressing the "average pickness" of p. The choice of r remains however quite arbitrary.
- $-\log(M_r(p))$ is a natural generalization of $H(p)$!

Deutsch; take no.2

- In 1961 Landau and Pollak showed that

$$\arccos C \leqslant \arccos M_\infty(p) + \arccos M_\infty(q)$$

which implies

$$M_\infty(p) M_\infty(q) \leqslant \tfrac{1}{4}(1+C)^2,$$

- This gives precisely the Deutsch result in the form

$$H(p) + H(q) \geqslant 2 \log \frac{2}{1+C}$$

- Is that the best we can do?... No!

Riesz

Riesz theorem (1926):

Let $\psi \in \mathbb{C}^N$ and $T : \mathbb{C}^N \to \mathbb{C}^N$ be linear map such that $\sum_k |T\psi_k|^2 = \sum_k |\psi_k|^2$. Then

$$c^{1/m}\left(\sum_k |T\psi_k|^m\right)^{1/m} \leqslant c^{1/n}\left(\sum_k |\psi_k|^n\right)^{1/n},$$

where $c = \max\{T_{kl}|k,l\}$ and $1/n + 1/m = 1$, $1 \leqslant n \leqslant 2$.

We implement that by taking

$$n = 2(1+r),\;\; m = 2(1+s)$$

$$\psi_l = \langle \psi, a_l \rangle$$

$$T_{kl} = \langle a_l, b_k \rangle$$

which yields $\quad T\psi_k = \sum_l \langle \psi, a_l \rangle \langle a_l, b_k \rangle = \langle \psi, b_k \rangle .$

Krauses result

- By substitution we get

$$M_r(p)\ M_s(q) \leqslant c^2$$

for $0 \leqslant s,\ r = -s/(2s+1)$ or the opposite.

- Taking $s = 0$, leads

$$H(p) + H(q) \geqslant -2\log c$$

– the result obtained by Kraus (1987)

Example:

$p, q =$ orthogonal spin projections $\Rightarrow c = 1/\sqrt{2} \Rightarrow -2\log c \approx 0.3$.

Remark:

In general $M_r(p)$'s are significant on their own! One can get much better for different r, s.

Mixed states

- Consider a general mixed state

$$W = \sum_k \alpha_k \langle \cdot, \psi_k \rangle \psi_k,$$

$$\sum_k \alpha_k = 1, \ \alpha_k \geqslant 0.$$

- Corresponding probabilities become

$$\bar{p}_k = \sum_l \alpha_l p_{lk}, \qquad \bar{q}_k = \sum_l \alpha_l q_{lk}$$

- Question:
  Is $M_r(\bar{p}) M_s(\bar{q}) \leqslant c^2$ still valid?

Mixed states

- By choosing r, s as above:

$$0 \leqslant s, \qquad r = -s/(2s+1)$$

we can get (Maassen & Uffnik 1988)

$$\left(\sum_k \bar{p}_k{}^{1+r}\right)^{1/(1+r)} \geqslant c^{2r/(r+1)} \left(\sum_k \bar{q}_k{}^{1+s}\right)^{1/(1+s)} .$$

- This yields

$$M_r(\bar{p})\ M_s(\bar{q}) \leqslant c^2 .$$

$\dim \mathscr{H} = \infty$

- In the case of infinite-dimensional Hilbert space the results above can be generalized!
  We take
  $$M_r(p) = \left(\int p(x)^{r+1} \, \mathrm{d}x \right)^{1/r}$$
  and
  $$H(p) = -\int p(x) \log p(x) \, \mathrm{d}x.$$
  However now $H(p)$ might have negative values!
  This generalization is not direct!
- For the case $q = |\psi|^2$, $p = |\hat{\psi}|^2$, $(\psi \in \mathscr{H} = L^2(\mathbb{R}))$ we can apply

The Hausdorff-Young inequality:

$$c^{1/m} \left(\int \left| \hat{\psi}(k) \right|^m \mathrm{d}k \right)^{1/m} = c^{1/n} \left(\int \left| \psi(x) \right|^n \, \mathrm{d}x \right)^{1/n},$$

where $c = 1/\sqrt{2\pi}$ and $1/n + 1/m = 1$, $1 \leqslant n \leqslant 2$.

Hirschman

- The same substitution

$$n = 2(1+r),\ \ m = 2(1+s)$$

leads to

$$M_r\left(|\psi|^2\right)\ M_s\left(|\hat{\psi}|^2\right) \leqslant \frac{1}{2\pi}$$

- And in case $r = s = 0$

$$H\left(|\psi|^2\right) + H\left(|\hat{\psi}|^2\right) \geqslant \log 2\pi$$

(Hirschman 1957)

Problem:

Not every $M_r(p)$ can be an uncertainty measure as some does not have minimum at $p(\delta)$!

Summary

- Robertson relations do not express true uncertainty principle.
- There are natural restrictions on the Shannon entropy.
- Generalizations to the infinite-dimensional cases are possible although come with difficulties.

Resources:

[1] D. Deutsch. *"Uncertainty in Quantum Measurements"*. Phys. Rev. Letters. (Feb 1983).

[2] H. Maassen, J. B. M. Uffnik. *"Generalized entropic uncertainty relations"*. Phys. Rev. Letters. (March 1988).

Special thanks to Konrad Szymański!

The end!