Classical Value of Non-Local Quantum Games

2019/05/06

Daniel Alsina (University of Leeds, UK)
WB (Jagiellonian University, Poland)
Dardo Goyeneche (Universidad de Antofagasta, Chile)
Ondřej Turek (University of Ostrava, Czech Republic)
Karol Życzkowski (Jagiellonian University, Poland)

table of contents

Bell theorem (inequality)

QM problems

is QM incomplete or not?

- Copenhagen interpretation of QM (excerpt)

  - wavefunction carries entire information about the system

  - measurement is disturbing and reveals actual values - before, there are only prob. distr.
- quantum entanglement - a "strange" property of composite systems
- Einstein-Podolski-Rosen (1935) - assuming Local Realism, QM is missing something...

  - photons' polarization is set at the very beginning

  - when separated they cannot "communicate" each other
- Local Hidden Variables theory (λ)

  - "spooky action at a distance"
- Bell's no-go theorem (1964)

  - LR implies inequalities which are violated in experiment
- conclusion: QM describes **non-local** Universe!
- **CHSH generalizations (Clauser, Horne, Shimony, Holt, 1969)**

CHSH (1969)

CHSH version of the Bell's inequality:

suppose "binary" experiment (possible outcomes are normalized to ∓ 1, $q = 2$)

$$|\mathbf{C}(a,b) + \mathbf{C}(a,b') + \mathbf{C}(a',b) - \mathbf{C}(a',b')| \leqslant 2$$

- a, a', b, b' - detector settings for Alice and Bob
- **C** - quantum correlation of the particle pairs

  eg.

$$\mathbf{C}(a,b) = \int_{\Lambda} AB\mu(\lambda)d\lambda \qquad \text{- derivation elsewhere...}$$

result:

- maximal classical value is 2
- maximal quantum value is $2\sqrt{2}$ (Tsirelson's bound)

  $\Longrightarrow$ **QM violates CHSH!**

---

violations confirmed in experiments

A. Aspect et al. Phys. Rev. Lett. **49** 1804 (1982)

excess of a matrix

excess of a matrix - introduction

Hadamard matrix = square $(1,-1)$-matrix H_n of order n with orthogonal rows (or columns); $H_n H_n^T = n\mathbb{I}_n$

equivalence relation: $H \simeq P_A D_A H D_B P_B$ for diagonals D_A, D_B with ∓ 1 and permutations P_A, P_B

- 1973, K.W. Schmidt asks for an estimate on the maximal number of ones in a Hadamard matrix of order n
- 1977, M.R. Best, "The Excess of a Hadamard matrix" - bounds for excess within a class of Hadamards of given size

excess of a matrix H_n (in quantum language):

$$\Sigma(H_n) \equiv \sum_{j,k=0}^{n-1} (H_n)_{jk} = n\langle\varphi_+|H_n|\varphi_+\rangle \quad : \quad |\varphi_+\rangle = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}|j\rangle$$

---

original Best's bounds for maximum excess (over the set $\mathcal{H}_n$ of all real Hadamards of size n):

$$n^2 2^{-n}\binom{n}{\frac{n}{2}} \leqslant \max_{\mathcal{H}_n}\left(\Sigma(H_n)\right) \leqslant n\sqrt{n} = \sigma(n)$$

(optimal) q-excess of a matrix

optimal excess of a Hadamard matrix H_n:

$$E(H_n) \equiv \max_{D_A, D_B} \Sigma(D_A H_n D_B) = \max_{D_A, D_B} n\langle\varphi_+|D_A H_n D_B|\varphi_+\rangle$$

optimization is taken over diagonal matrices D_A, D_B having ∓ 1 entries (square roots of unity)

generalized optimal excess (q-excess) of a complex matrix M:

$$E_q(M) \equiv \max_{D_A, D_B} \mathrm{Re}\Big(\Sigma\left(D_A M D_B\right)\Big)$$

optimization is taken over diagonal matrices D_A, D_B having q^{th} roots of unity on the diagonal, eg. $(D_A)_{jj} = {\omega_q}^{f_j}$ for $\omega_q = \exp 2\pi i/q$ and $f_j \in \mathbb{Z}$

remarks:

- Σ is standard excess
- $E_2(H_n) = E(H_n)$
- $\Sigma(H_n) \leqslant E(H_n) \leqslant \sigma(n)$
- q has a physical interpretation ($q < \infty$)

CHSH - connection with q-excess

simplest example of the **Bell** operator for **CHSH**

$$\mathbf{B} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$$

classical value

$$C = \max_{a_j, b_k = \mp 1} \left(+ a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \right)$$

$$C = \max_{a_j, b_k = \mp 1} \left(H_{00} a_0 b_0 + H_{01} a_0 b_1 + H_{10} a_1 b_0 + H_{11} a_1 b_1 \right)$$

$$C = \max_{a_j, b_k = \mp 1} \sum \left[\begin{array}{cc} H_{00} a_0 b_0 & H_{01} a_0 b_1 \\ H_{10} a_1 b_0 & H_{11} a_1 b_1 \end{array} \right] =$$

$$= E_2 \left(\left[\begin{smallmatrix} + & + \\ + & - \end{smallmatrix} \right] \right) = E_2(H_2)$$

CHSH - connection with q-excess

simplest example of the **Bell** operator for **CHSH**

$$\mathbf{B} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$$

classical value

$$C = \max_{a_j, b_k = \mp 1} \left(+ a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \right)$$

$$C = \max_{a_j, b_k = \mp 1} \left(H_{00} a_0 b_0 + H_{01} a_0 b_1 + H_{10} a_1 b_0 + H_{11} a_1 b_1 \right)$$

$$C = \max_{a_j, b_k = \mp 1} \sum \begin{bmatrix} H_{00} a_0 b_0 & H_{01} a_0 b_1 \\ H_{10} a_1 b_0 & H_{11} a_1 b_1 \end{bmatrix} =$$

$$= E_2 \left(\left[\begin{smallmatrix} + & + \\ + & - \end{smallmatrix} \right] \right) = E_2(H_2) \to E_q(M)$$

generalization: other (complex) Hadamards, unitary matrices, orthogonal designs, etc...

CHSH - connection with q-excess

scenario: 2 parties, n settings and q outcomes

in this model no further generalisation to more parties is possible

M - an $n \times n$ matrix

(optimal) q-excess $E_q(M)$ (value) corresponds to a classical (C) value of the bipartite Bell inequality associated with M

- **(optimal) extended q-excess** Ξ_q (operator) coincides with the Bell operator generated by M
- **(optimal) extended q-excess** $\mathcal{E}_q^d$ (value) determines the maximal violation (Q) of a bipartite Bell inequality allowed in quantum theory

`mathematical paper...`

non-local quantum games (with no quantum advantage)

non-local quantum game

- two (or more) parties and referee (all separated)
- pre-shared entangled state $|\phi\rangle$ and common strategy
- no classical communication
- parties are asked by referee and referee makes (calculates) decision

aim: joint winning

$G = G(\rho, V)$

- $\rho(x, y)$ - probability distribution on the set of "questions" $\mathbb{A} \times \mathbb{B}$
- $V(a, b; x, y)$ - pay-off function

$$\omega_C(G) \equiv \max_{a,b} \sum_{x,y} \rho(x, y) V(a, b; x, y) \mathbb{P}(a, b; x, y)$$

$$\omega_Q(G) \equiv \max_{|\phi\rangle, A_a^x, B_b^y} \sum_{x,y} \rho(x, y) V(a, b; x, y) \langle\phi| A_a^x \otimes B_b^y |\phi\rangle$$

XOR game

subclass: XOR games $\subset$ NLQGames

$a \oplus b = f(x, y)$

with a deterministic function $f(x, y)$ based on questions (inputs)

R. Ramanathan et al. Phys. Rev. Lett. **113**, 240401 (2014)

---

applications in QIT

- self-testing
- quantum cryptographic protocols
- (device independent) quantum key distribution

D. Meyers, A. Yao, "Self testing quantum apparatus", QIC **4**, 4, 273-286 (2004)

XOR game example - classical case

CHSH game example

A and B wins the game iff $a \oplus b = xy$

- if either $x = 0$ or $y = 0$ then $a = b$
- $x = y = 1$ then $a \neq b$

strategy: both answer 0

x	y	a	b	$a \oplus b = xy$
0	0	0	0	$0 = 0$
0	1	0	0	$0 = 0$
1	0	0	0	$0 = 0$
1	1	0	0	$0 \neq 1$(miss!)

$$\mathbb{P}(\text{A and B win}) = 3/4$$

Scott Aaronson's blog • https://www.scottaaronson.com/qclec/14.pdf

XOR game example - quantum case

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - \text{entanglement!}$$

WLOG: A takes the 1^{st} qubit from the quantum system and performs 1^{st} measurement

strategy:

- $x = 0 \implies A$ measures her qubit in $\{|0\rangle, |1\rangle\}$
- $x = 1 \implies A$ measures her qubit in $\{|-\rangle, |+\rangle\}$
- $y = 0 \implies B$ measures his qubit in $\{|0\rangle, |1\rangle\}$ rotated by $-\pi/8$
- $y = 1 \implies B$ measures his qubit in $\{|0\rangle, |1\rangle\}$ rotated by $\pi/8$

output: $a = 0$ when A measures $|0\rangle$ or $|+\rangle$ and $a = 1$ otherwise

$b = 0$ when B measures $|\pi/8\rangle$ or $|-\pi/8\rangle$ and $b = 1$ otherwise

x	y	$\mathbb{P}$(same output)
0	0	$\cos^2 \pi/8$
0	1	$\cos^2 \pi/8$
1	0	$\cos^2 \pi/8$
1	1	$1 - \cos^2 3\pi/8 \to \cos^2 \pi/8$

$$\implies \mathbb{P}(\text{both win}) = \frac{2+\sqrt{2}}{4} \approx .85 > .75$$

quantum advantage

this was the example of game in which parties can use quantum resources to achieve a better result than in the classical world

quantum advantage

nowadays the NLQG for which there is NO QUANTUM ADVANTAGE seems to be equally interesting as the ones with QA as they are used to work out the limitations of QM

GYNI game (no QA)

Guess Your Neighbour Input game - a multipartite non-local quantum game with no QA

there are N players and

- each player receives one bit x_i
- each player outputs one bit a_i
- everyone must guess its (left) neighbour's input bit : $a_i = x_{i+1 \bmod N}$

rules

- no communication
- sharing physical system and/or strategy
- $\rho(x_1, ..., x_N) = \rho(\mathbf{x})$ is known

(average) winning probability

$$\omega = \sum_{\mathbf{x}} \rho(\mathbf{x}) \mathbb{P}(\mathbf{a}_i = \mathbf{x}_{i+1} | \mathbf{x})$$

with $\mathbb{P}(\mathbf{a}_i = \mathbf{x}_{i+1} | \mathbf{x}) = \mathbb{P}(a_1 = x_2, ..., a_n = x_1 | x_1, ..., x_n)$

GYNI game - example

there exists a winning strategy in classical and quantum case with no QA

L. Almeida et al. Phys. Rev. Lett. **104**, 230404 (2010)

A. Acín et al. "Quantum Theory: Informational Foundations and Foils" (2012)

---

particular version of the 3-partite GYNI game with $x_1 \oplus x_2 \oplus x_3 = 0$

then

$$\omega = \frac{1}{4}\Big(\mathbb{P}(000|000) + \mathbb{P}(110|011) + \mathbb{P}(011|101) + \mathbb{P}(101|110)\Big)$$

stay tuned...

regular matrix

definition:

A matrix H is said to be **regular** iff all row entries sum up equally.

---

for Hadamard (or normal $[N, N^*] = 0$) matrices one takes either rows or columns

H_n - regular Hadamard matrix $(\sqrt{n} \in \mathbb{N}) \Longrightarrow E(H_n) = \sigma(n) = n\sqrt{n}$

Frobenius-Perron theorem: for any matrix $M = (m_{jk})$ of positive entries one has (among other properties)

- $\exists\, r \in \mathrm{eig}(M) : r > 0$ such that $\forall\, \lambda \in \mathrm{eig}(\mathrm{M}) : r > |\lambda|$
- $\min_j \sum_k m_{jk} \leqslant r \leqslant \max_j \sum_k m_{jk}$

results

series of facts - (1)

1. The **classical** value of a bipartite Bell inequality having n settings and q outcomes, induced by a matrix M, coincides with the optimized q-excess $E_q(M) = \max_{D_A, D_B} \mathrm{Re}\Big(\Sigma(D_A M D_B)\Big)$

---

remarks:

- optimization problem has been solved for many classes of matrices (Hadamard, orthogonal designs...)
- classical BI computation is `NP-complete` (quantum - `NP-hard`)
- real-valued matrices vs. complex-valued matrices...
  $\rightarrow$ `mathematical paper`

observation 2 and 3

2. For any matrix M of order n and any $q > 1$ one has $E_q(M) \leqslant n\sigma_{\max}(M)$.

3. For any **regular** matrix M of order n with **positive** entries and for any $q > 1$ one has $E_q(M) = n\sigma_{\max}(M)$.

in both cases $\sigma_{\max}(M)$ is the biggest singular value of M

---

remarks:

- in 2. equality holds true for Hadamard matrices only
- in 3. one uses FP theorem

observation 4 (!)

4. Any **bipartite** quantum game

$$C(M) = \max_{a_x, b_y = \mp 1} \sum_{x,y=0}^{n-1} M_{x,y} a_x b_y$$

associated to a **regular** matrix M of order n, for any number of settings (n) and any number of outcomes q, has a classical value given by the sum of the n^2 entries of M.

Furthermore there is **no quantum advantage**

$$\omega_C(M) = \omega_Q(M).$$

---

remarks:

- improving recent results
- extension/simplification of some problems with known classes of NLQG
- the presence of QA does not depend on eigenvalues/-vectors of M

also...

- "elegant" Bell inequalities and other specific classes of BI
- link between contextuality and non-locality and "tight BI"
- graph theory and...
- ...extension of the set of zero-error Shannon capacity channels
- multidimensional tensors
- ...

3-partite GYNI game revisited

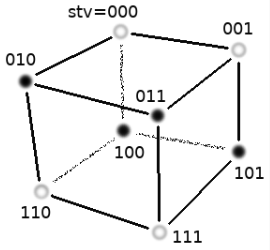
...

$$\omega = \frac{1}{4}\Big(\mathbb{P}(000|000) + \mathbb{P}(110|011) + \mathbb{P}(011|101) + \mathbb{P}(101|110)\Big)$$

one can associate the following tensor:

$$M^{stv}_{000} = 1; \quad M^{stv}_{011} = (-1)^{s+t}; \quad M^{stv}_{101} = (-1)^{t+v}; \quad M^{stv}_{110} = (-1)^{s+v}$$

where $s, t, v = 0, 1$ and each of these tensors defines a **regular** Hadamard cube, eg.



summary

- one-to-one connection between the notion of excess and the classical value of the related Bell inequality

  one can examine infinitely many inequalities defined by (complex) Hadamard matrices (Butson-type matrices), orthogonal designs, etc...

- **(optimal) q-excess** $E_q(M)$ corresponds to the classical value of the bipartite Bell inequality associated with matrix M of size n

  2 parties, n settings and q outcomes

- **(optimal) extended q-excess** $\Xi_q(M)$ determines a bipartite Bell operator for any (rectangle) matrix M
- **(optimal) extended q-excess** $\mathcal{E}_q^d(M)$ corresponds to the quantum value for the bipartite Bell inequality associated with M ($d = q$)

  in general $E_q(M) \leqslant \mathcal{E}_q^d(M)$

- eigen-analysis of a matrix does not provide any information about quantum advantage
- regular matrices define non-local quantum games with no QA
- ...

additional references

R. M. Best, Ind. Math. **80**, 5, 357-361 (1977)

N. Farmakis, S. Kounias, Discr. Math. **67**, 2, 165-176 (1987)

H. Kharaghani, J. Seberry, Graphs and Comb. **9**, 1, 47-56 (1993)

J.F. Clauser et al., Phys. Rev. Lett. **23**, 880 (1969)

B.S. Tsirelson, Lett. Math. Phys. **4**, 93 (1980)

J. Batle, et al., Quant. Inf. Proc. **15**, 6, 2649 (2016)

U. Vasirani, T. Vidick, Phys. Rev. Lett. **113**, 140501 (2014)

J. Bancal et al., J. Phys. A: Math. Theor. **43**, 385303 (2010)

A. Cabello et al., `arXiv:1010.2163` (2010)

C.E. Shannon, IRE Trans. Inform. Th. **2**, 8-19 (1956)

L. Lovász, IEEE Trans. on Inform. Theory, IT-25 (1) (1979)

A. Acín et al., Phys. Rev. **A93**, 040102(R) (2016)

Thank You