Classical limit of entangled states of two angular momenta

Jan Mostowski
(based on a paper with
Marek Kuś
Joanna Pietraszewicz)

Institute of Physics, Polish Academy of Sciences

Kraków, April 15, 2019

Introduction

Correspondence principle – large (highly excited) quantum system should exhibit classical features.

Routes to classical limit:

- interactions with environment – "dephasing" to a superposition of "pointer" states
- discuss quantities that have classical counterparts
- Perform very precise measurements to detect quantum effects

This talk – classical limit of entanglement

Beginnig – two level system

Consider a spin (two level system), two orthogonal states levels are denoted by $|u\rangle$ and $|v\rangle$.
Transformation of these states – rotation of axes:

$$|u'\rangle = \cos(\alpha/2)|u\rangle + \sin(\alpha/2)|v\rangle$$

$$|v'\rangle = -\sin(\alpha/2)|u\rangle + \cos(\alpha/2)|v\rangle$$

for any value of α.

Two particle system

Consider now a system of two separated spins.

States are (tensor) products of both subsystem states, and linear combinations of such states.

Choose the following state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|u\rangle|v\rangle - |v\rangle|u\rangle)$$

This is a singlet state, i.e. total spin is zero. It has the same form in any coordinate system.

This two particle state is entangled. Reduced density matrix of the first subsystem $\rho(w, w') = \frac{1}{2}\delta_{w,w'}$ is diagonal and its entropy is maximal – maximal entanglement.

Correlation functions and Bell's inequality

Define correlation function in the state Ψ

$$E(\mathbf{a},\mathbf{b}) = \langle(\mathbf{a}\cdot\mathbf{s_1})(\mathbf{b}\cdot\mathbf{s_2})\rangle$$

.

Introduce two pairs of vectors:

$\mathbf{a}$ and $\mathbf{a}'$ for observer A,
$\mathbf{b}$ and $\mathbf{b}'$ for observer B.

Choose $\mathbf{aa}' = 0$, $\mathbf{bb}' = 0$, $\mathbf{ab} = \frac{1}{\sqrt{2}}$, $\mathbf{ab}' = -\frac{1}{\sqrt{2}}$.
Bell's inequality for correlation functions

$$\mathbf{B} = E(\mathbf{a},\mathbf{b}) + E(\mathbf{a}',\mathbf{b}') + E(\mathbf{a}',\mathbf{b}) - E(\mathbf{a},\mathbf{b}') \leqslant 1$$

Quantum calculation gives $\mathbf{B} = \sqrt{2} > 1$

Violation of Bell's inequality.

Bell's inequalities – probabilities

Classical probability theory – spin represented as classical dichotomic variable, values $\pm\frac{1}{2}$.
Let $p(\mathbf{a}, r)$ – probability that the spin value is r along $\mathbf{a}$,
Similar definition for the second spin and other directions.

Let $p(\mathbf{a}, \mathbf{b}, r, s)$, etc. – joined probability of finding value r of one spin and value s of second spin along directions $\mathbf{a}$ and $\mathbf{b}$.
The following inequality should hold if r and were classical dichotomic variables:

$$P = p(\mathbf{a}, \mathbf{b}, u, v) + p(\mathbf{a}', \mathbf{b}, u, v) + p(\mathbf{a}', \mathbf{b}', u, v) - p(\mathbf{a}, \mathbf{b}', u, v) \leqslant 1$$

Quantum mechanical calculation gives $P = \frac{1}{2}(1 + \sqrt{2})$.
Violation of another Bell's inequality.

Quantum versus classical

Classical probabilities – satisfy Bell's inequality

$$P \leqslant \frac{1}{2}(1+\sqrt{2})$$

rather than $\leqslant \frac{1}{2}$.
Quantum probabilities – violate this inequality.
Also quantum correlation function violates another Bell's inequality.

$$E(\mathbf{a},\mathbf{b}) + E(\mathbf{a}',\mathbf{b}') + E(\mathbf{a}',\mathbf{b}) - E(\mathbf{a},\mathbf{b}') \leqslant \sqrt{2}$$

rather than $\leqslant 1$.

Where is the Planck constant $\hbar$?

Generalizations – large angular momentum

We will now consider an analogous system with large angular momentum j described by state

$$\Psi\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m} |m\rangle | -m\rangle$$

Total angular momentum is zero.
Regardless of the value of j the state $|\Psi\rangle$ is maximally entangled.
Can one prove in experiments that entanglement in this state really exists?

Classical system

Classical counterpart will be considered before further study of the quantum system.

Two spatially spinning rotors, each with angular momentum J,
total angular momentum – zero.

Let $\mathbf{n}$ be direction of angular momentum of one particle, hence

$$\mathbf{J}_1 = \mathbf{n}J,\ \mathbf{J}_2 = -\mathbf{n}J$$

Consider $\mathbf{n}$ as a random direction with uniform distribution.

This classical system is the classical limit of the quantum state $|\Psi\rangle$.

Classical measurements

We now find the probability density $R(\mathbf{a},\mathbf{b},K,L)$ of measuring angular momentum of first particle along axis $\mathbf{a}$ to be equal K and of second angular momentum along axis $\mathbf{b}$ to be equal to L.

$$R(\mathbf{a},\mathbf{b},K,L) = \frac{1}{4\pi}\int d\Phi \, \sin\Theta \, d\Theta \;\; \delta(K - J\mathbf{a}\,\mathbf{n}) \cdot \delta(L + J\mathbf{b}\,\mathbf{n}).$$

Introduce scaled angular momenta $k = K/J$ and $l = L/J$. The probability for finding the quantities k and l within the range of dk and dl is $p_c(\mathbf{a},\mathbf{b},k,l) = \rho(\mathbf{a},\mathbf{b},k,l)\,dk\,dl$.
After integrating we find:

$$\rho(\mathbf{a},\mathbf{b},k,l) = \frac{1}{2\pi}(-k^2 - l^2 - 2\,k\,l\cos\theta + \sin^2\theta)^{-\frac{1}{2}}$$

Correlation functions

We can now find the classical and quantum correlation function of $\mathbf{a} \cdot \mathbf{J}_1$ and $\mathbf{b} \cdot \mathbf{J}_2$. In the classical state we find:

$$E(\mathbf{a}\,\mathbf{J}_1, \mathbf{b}\,\mathbf{J}_2) = -\frac{j^2}{3}\,\mathbf{a} \cdot \mathbf{b}$$

In the quantum case, in the state $|\Psi\rangle$. we find:

$$E(\mathbf{a}\,\mathbf{J}_1, \mathbf{b}\,\mathbf{J}_2) = -\frac{j(j+1)}{3}\,\mathbf{a} \cdot \mathbf{b}$$

Notice, that quantum and classical correlations are almost the same for large j.

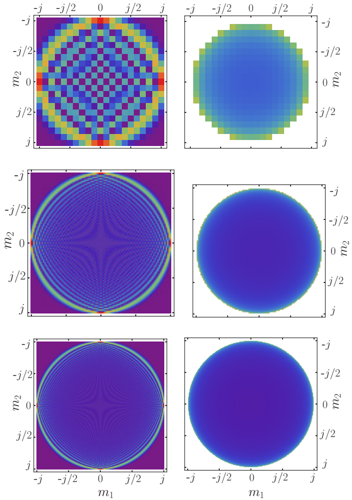
No way to distinguish between quantum and classical case on the basis of correlation functions.

Probability amplitude

We will find probability amplitude $a(\mathbf{a}, \mathbf{b}, m_1, m_2)$ of detecting values m_1 and m_2 (in the chosen coordinate system):

$$a(\mathbf{a}, \mathbf{b}, m_1, m_2) = \frac{1}{\sqrt{2j+1}} (-1)^{j-m_1} \, \mathbf{d}^j_{-m_1, m_2}(\beta). \tag{1}$$

where $\mathbf{d}^j_{-m,m'}(\beta)$ denotes the Wigner rotation function, β is the angle between $\mathbf{a}$ and $\mathbf{b}$.



Rysunek: Probabilities: quantum (left) and classical (right) for $\beta = \pi/2$ and $j = 10$, $j = 50$, $j = 90$ for upper, middle and low panels.

Bell's inequalities for large j states

We will now consider Bell's inequalities for probabilities.
If hidden variables exist and local realism is valid then probabilities $p(\mathbf{a}_r, m)$ and $p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ are within the convex hull spanned by vertices defined by these functions $\longrightarrow$ Bell's inequalities.

Hard to find explicit form of the convex hull.

In what follows we look for violation of these inequalities.

Semiclassical approximation

Use semiclassical approximation to find relations between quantum and classical probabilities.
Probability amplitudes – the Wigner rotation function.

$$\text{Approximation to } \mathbf{d}^{j}_{m,m'}(\beta) \text{ is needed.}$$

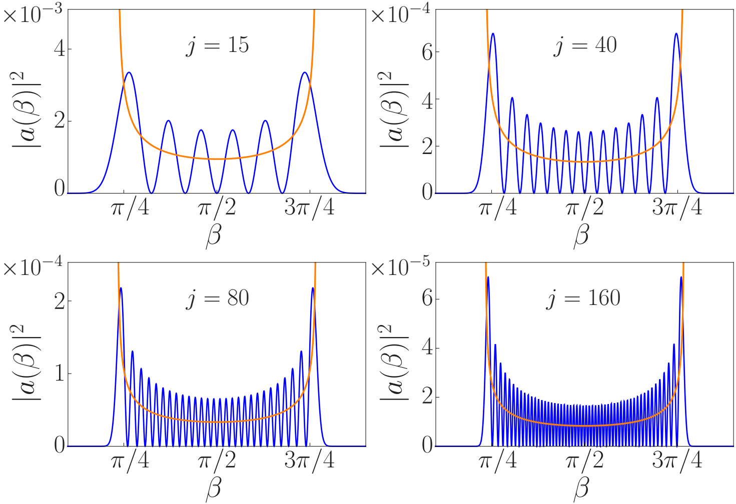
Some results – in Wigner's book on the rotation group.
Exercise in semiclassical (WKB) method (Fritz Haake and coworkers).

Semiclassical approximation – formula

$$\mathbf{d}^j_{m,m'}(\beta) = (-1)^j \cos\left(\bar{J} S_0(\beta) - \frac{\pi}{4}\right) \times$$
$$\times \left(\frac{2}{\pi\sqrt{\sin^2\beta - \frac{1}{\bar{J}^2}\left(m^2 + m'^2 - 2\, m\, m'\ \cos\beta\right)}}\right)^{\frac{1}{2}}.$$

$$\begin{aligned} S_0\left(m', m\right) &= \frac{m}{\bar{J}} \arccos\left(\frac{m\cos\beta - m'}{\sin\beta\sqrt{\bar{J}^2 - m^2}}\right) \\ &- \frac{m'}{\bar{J}} \arccos\left(\frac{m - m'\cos\beta}{\sin\beta\sqrt{\bar{J}^2 - m'^2}}\right) \\ &+ \arccos\left(\frac{m\, m' - \bar{J}^2\cos\beta}{\sqrt{\left(\bar{J}^2 - m'^2\right)\left(\bar{J}^2 - m^2\right)}}\right) \end{aligned}$$

Exact versus semiclassical



Rysunek: Probability distributions as a function of the projected angle β

Generalized Bell's inequalities

There are $S = 4 \times (2j+1)^2$ inequalities (not all are independent), all of the form

$$M_1 \leqslant \sum_{r,s,\mu,\nu} p(\mathbf{a}_r, \mathbf{b}_s, m_\mu, m_\nu)\, c(r,s,m_\mu,m_\nu) \leqslant M_2,$$

where M_1 and M_2 are fixed numbers and $c(r,s,m_\mu,m_\nu)$ are coefficients.

The coefficients are equal to 0 or to natural numbers with plus or minus sign.

The number of independent inequalities grows rapidly with j. In the case of large j the number of Bell's inequalities is huge. It is not realistic to consider them all, so we will turn to statistical approach.

Inspiration

Inspiration – from a somewhat similar approach to spectra of complex systems.

Hamilton operator is replaced by a matrix with elements being random numbers with a statistical distribution. The eigenvalues are random numbers and are studied using statistical methods.

Also other systems are studied with statistical methods.

Quantum corrections

Probabilities of measuring given values of angular – random numbers.

It is helpful to consider quantum corrections, i.e. differences between the quantum and classical probabilities:

$$\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) = p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) - p_c(\mathbf{a}_r, \mathbf{b}_s, k, l)dkdl, \quad (2)$$

where $k = m_1/j$, $l = m_2/j$ and increments are chosen to fit the increments of quantum numbers m_1 and m_2, therefore $dk = 1/j = dl$.

Classical probabilities satisfy Bell's inequalities.

Violation of these inequalities by quantum systems can be due to the quantum corrections only.

Dependence of quantum corrections on j

Scaling of the quantum corrections with the value of angular momentum j. Observe that

$$\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \rightarrow 0$$

for $j \rightarrow \infty$, (regardless of the choice of vectors $\mathbf{a}_r$ and $\mathbf{b}_s$).

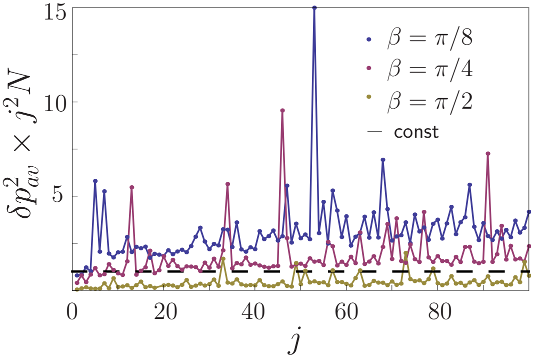
Introduce average values of corrections, i.e.

$$\delta p_{av} \;=\; \frac{1}{(2j+1)^2} \sum_{m_1, m_2} \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$$

.

Notice that $\delta p_{av} \rightarrow 0$ when $j \rightarrow \infty$.

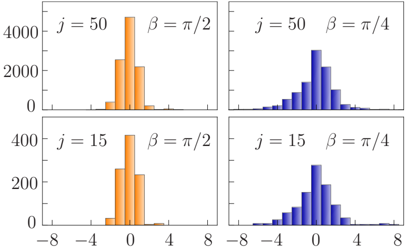
Scaling with j



Rysunek: The scaling of average squares of $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ versus momentum j. Notice that δp_{av}^2 goes to zero as j^{-4}.

Histograms of probabilities

Histograms of differences between classical and quantum probabilities. Differences are random numbers with a narrow distribution.



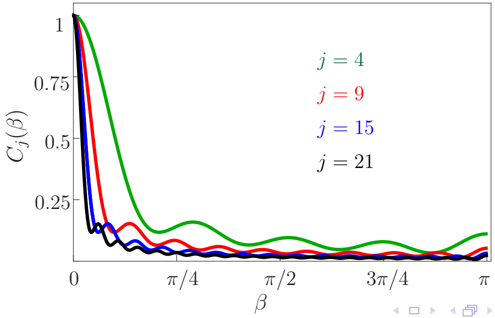
Rysunek: Histograms of $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \times N$ for many values of projections of momentum $m_{1,2}$ and angle β.

Correlation between probabilities

Probabilities $p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ for various angle between $\mathbf{a}_r$ and $\mathbf{b}_s$ are not correlated. Correlation function:

$$C_j(\beta) = N \sum_{m_1, m_2 = -j}^{j} [p(\mathbf{a}_r, \mathbf{a}_s, m_1, m_2) \cdot p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)]$$

decreases rapidly with j in the vicinity of small angles.



Scaling

Bell's inequalities – reformulated for $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$.
Each contains $(2j+1)^2$ terms at most, their number scales as j^2.
Each term – product of a coefficient $c(r, s, m_1, m_2)$ and $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$.
The number of coefficients – proportional to j^2, while $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ – to j^{-2}.
Linear combination tends to 0 or to a constant, independent of j.
$\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ – are positive or negative, the sum tends to 0 for $j \to \infty$.
Exception – when signs of $\delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ are correlated with the signs of $c(r, s, m_1, m_2)$.

Sensitivity to parameters

Probabilities $p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ are very sensitive to the angle between the two vectors.
Suppose that one of Bell's inequalities is violated by a set of probabilities for given vectors $\mathbf{a}_r$ and $\mathbf{b}_s$.

A small change of angle between them, by about π/j leads to a substantial change of the probabilities and the inequality is valid.

Verification if a Bell inequality is violated or not, depends on the choice of angles, and therefore on the accuracy of angle determination.

Conclusions – reaching classical limit

- Coefficients $c(r, s, m_1, m_2)$ have to be correlated with the probabilities $p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)$ to strongly violate a Bell inequality.
- Some Bell's inequalities can be violated by probabilities obtained form the state $|\Psi\rangle$.
- On average the violation of Bell's inequalities is negligible.
- Degree of practical violation does not grow with j.
- The quantum state, $|\Psi\rangle$, reproduces classical angular momentum in spite of being maximally entangled.
- Classical limit proved here is not reached in a uniform way.

THE END

THANK YOU

Probability amplitudes – quantum approach

Two observers, A and B, measure components of angular momenta along arbitrarily chosen axes in the state $|\Psi\rangle$.

We name these axes $\mathbf{a}$ and $\mathbf{a}'$ for observer A and $\mathbf{b}$ and $\mathbf{b}'$ for observer B.
The probability of finding the value $s = \pm\frac{1}{2}$ of the spin by each observer along each axis is s-independent and flat $\frac{1}{2}$.
This is equal to

Quantum correlation functions

In addition to the probability amplitude $a(\mathbf{a}, \mathbf{b}, m_1, m_2)$ we consider correlation function $\mathbf{a}\,\mathbf{J}_1 \cdot \mathbf{b}\,\mathbf{J}_2$ in the state $|\Psi\rangle$. We find:

$$E(\mathbf{a}\,\mathbf{J}_1, \mathbf{b}\,\mathbf{J}_2) = \sum_{m=-j}^{j} \frac{1}{2j+1} m(-m)\cos(\beta). \tag{3}$$

where β is the angle between $\mathbf{a}$ and $\mathbf{b}$.
The sum over m is well known. In general, we can write:

$$E(\mathbf{a}\,\mathbf{J}_1, \mathbf{b}\,\mathbf{J}_2) = -\frac{j(j+1)}{3}\,\mathbf{a} \cdot \mathbf{b}$$

This is consistent with the spin $\frac{1}{2}$ system.

Measurements of angular momenta

Two observers, A and B, measure components of angular momenta along arbitrarily chosen axes in the state $|\Psi\rangle$.
The result of such measurement is a number m_1 in case of observer A and m_2 in case of observer B, with $-j \leqslant m_{1,2} \leqslant j$.

There are $2j+1$ possible outcomes of measurement for each observer.

Result of a measurement

The state $|\Psi\rangle$ is invariant under rotations, i.e. it has the same form in all coordinate systems.

Choose the system of coordinates:

vector $\mathbf{a}$ is along the z axis
vector $\mathbf{b}$ lies in the $x - z$ plane.

The probability of finding the value m of the angular momentum by each observer along each axis is m-independent and flat $\frac{1}{2j+1}$.

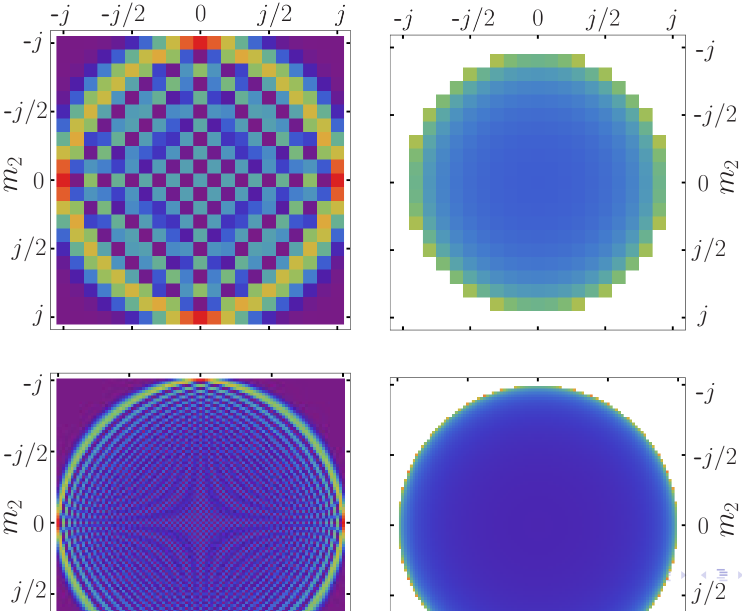
Probability amplitude

After short calculation – the probability amplitude $a(\mathbf{a}, \mathbf{b}, m_1, m_2)$ of detecting values m_1 and m_2 in the chosen coordinate system is:

$$a(\mathbf{a}, \mathbf{b}, m_1, m_2) = \frac{1}{\sqrt{2j+1}} (-1)^{j-m_1} \, \mathbf{d}^{j}_{-m_1, m_2}(\beta). \qquad (4)$$

where $\mathbf{d}^{j}_{-m,m'}(\beta)$ denotes the Wigner rotation function, β is the angle between $\mathbf{a}$ and $\mathbf{b}$.

graphics



Bell's inequalities

Bell's inequalities show the difference between quantum entangled states and classical correlations

Probability amplitude $a(\mathbf{a}, \mathbf{b}, m_1, m_2)$ of detecting a value m_1 of the angular momentum of the first particle in the direction $\mathbf{a}$ and a value of m_2 of the angular momentum of the second particle in the direction $\mathbf{b}$

$$a(\mathbf{a}, \mathbf{b}, m_1, m_2) = \frac{1}{\sqrt{2}}(-1)^{1/2-m_1}\ \mathbf{d}^{1/2}_{-m_1,m_2}(\beta). \tag{5}$$

One of these inequalities:

$$wstaw$$

is violated by the quantum expression

$$wstaw$$