Quantum Dynamical Entropy of Multiqubit Systems

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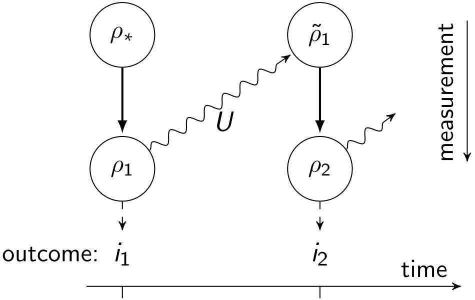
Successive measurements on a d-dim quantum system:

- projective measurement with d possible outcomes,
- unitary evolution U between two subsequent measurements.

- This procedure generates strings of random outcomes.
- Randomness is quantified by dynamical entropy.
- Chaotic unitaries: those with maximal dynamical entropy.



- Measurement: rank-1 PVM
  i.e., a set $\{\Pi_1, \dots, \Pi_d\}$ of 1-dim projectors on $\mathbb{C}^d$ such that $\sum_{i=1}^d \Pi_i = \mathbb{I}_d$
  $\Rightarrow$ $\Pi_j = |\varphi_i\rangle\langle\varphi_i|$ for $(\varphi_i)_{i=1,\dots,d}$ an orthonormal basis of $\mathbb{C}^d$.

- Probability of measuring $j \in \{1, \dots, d\}$ for the input state $\rho \in \mathcal{S}(\mathbb{C}^d)$:
  $$p_j(\rho) := \mathrm{tr}(\rho\,\Pi_j) = \langle\varphi_j|\rho|\varphi_j\rangle;$$
  in particular: $p_j(|\psi\rangle\langle\psi|) = |\langle\varphi_j|\psi\rangle|^2$

- State transformation if we measured j: $\quad \rho \longmapsto |\varphi_j\rangle\langle\varphi_j|$

- Evolution of the system between consecutive measurements:
  $$\mathcal{S}(\mathbb{C}^d) \ni \rho \mapsto U\rho\,U^\dagger \in \mathcal{S}(\mathbb{C}^d), \quad U : \mathbb{C}^d \to \mathbb{C}^d \text{ is unitary}$$

$$\begin{aligned} P_{i_1,\dots,i_n}(\rho_*) &= \text{probability of measuring } (i_1,\dots,i_n) \text{ starting from } \rho_* = \mathbb{I}/d \\ &= p_{i_1}(\rho_*) \cdot p_{i_1,i_2} \cdot \ldots \cdot p_{i_{n-1},i_n}, \end{aligned}$$

where $p_j(\rho_*) = \frac{1}{d}$ and $p_{jk} = |\langle \varphi_k | U | \varphi_j \rangle|^2$ for $j, k = 1, \dots, d$.

$$
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P_{i_1,\dots,i_n}(\rho_*) &= \text{probability of measuring } (i_1,\dots,i_n) \text{ starting from } \rho_* = \mathbb{I}/d \\
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\end{aligned}
$$

where $p_j(\rho_*) = \frac{1}{d}$ and $p_{jk} = |\langle\varphi_k| U |\varphi_j\rangle|^2$ for $j, k = 1, \dots, d$.

Partial entropy:

$$
H_n := \sum_{i_1,\dots,i_n=1}^{d} \eta\left(P_{i_1,\dots,i_n}(\rho_*)\right) \quad \text{with} \quad \eta(x) = \begin{cases} -x\ln x & x > 0 \\ 0 & x = 0 \end{cases}
$$

$$P_{i_1,\ldots,i_n}(\rho_*) = \text{probability of measuring } (i_1,\ldots,i_n) \text{ starting from } \rho_* = \mathbb{I}/d$$
$$= p_{i_1}(\rho_*) \cdot p_{i_1,i_2} \cdot \ldots \cdot p_{i_{n-1},i_n},$$

where $p_j(\rho_*) = \frac{1}{d}$ and $p_{jk} = |\langle\varphi_k| U |\varphi_j\rangle|^2$ for $j,k = 1,\ldots,d$.

Partial entropy:

$$H_n := \sum_{i_1,\ldots,i_n=1}^{d} \eta\left(P_{i_1,\ldots,i_n}(\rho_*)\right) \quad \text{with} \quad \eta(x) = \begin{cases} -x\ln x & x>0 \\ 0 & x=0 \end{cases}$$

Quantum entropy of U with respect to Π:

$$H(U,\Pi) := \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} (H_{n+1} - H_n)$$

For rank-1 PVMs: $$H(U,\Pi) = \frac{1}{d} \sum_{i,j=1}^{d} \eta\left(|\langle\varphi_i| U |\varphi_j\rangle|^2\right)$$

Quantum dynamical entropy of U (independent of measurement):

$$H^{dyn}(U) := \max\{H(U,\Pi) : \Pi \text{ is a rank-1 PVM}\}$$
$$= \max_{\substack{(\varphi_j)_{j=1}^d \\ \text{orthonormal} \\ \text{bases of } \mathbb{C}^d}} \frac{1}{d} \sum_{i,j=1}^{d} \eta\left(|\langle \varphi_i | U | \varphi_j \rangle|^2 \right)$$

- $H^{\mathrm{dyn}}(U)$ measures the ability of U to produce random outcomes;
- $H^{\mathrm{dyn}}(U)$ depends only on the eigenvalues of U;
- $0 \le H^{dyn}(U) \le \ln d$.

U is called **chaotic** iff $H^{\mathrm{dyn}}(U) = \ln d$

chaotic unitaries $\leftrightsquigarrow$ complex Hadamards

The following conditions are equivalent:

1. U is **chaotic**;
2. there exists an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of $\mathbb{C}^d$ such that
$$\frac{1}{d}\sum_{i,j=1}^{d}\eta\left(|\langle\varphi_i|U|\varphi_j\rangle|^2\right)=\ln d;$$
3. there exists an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of $\mathbb{C}^d$ in which $\sqrt{d}\,U$ is represented by a **complex Hadamard matrix**, i.e.,
$$|\langle\varphi_i|U|\varphi_j\rangle|=\frac{1}{\sqrt{d}}\quad\text{for each}\quad i,j=1,\ldots,d;$$
4. there exists an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of $\mathbb{C}^d$ such that $\{\varphi_i\}_{i=1}^d$ and $\{U\varphi_i\}_{i=1}^d$ are **mutually unbiased**.

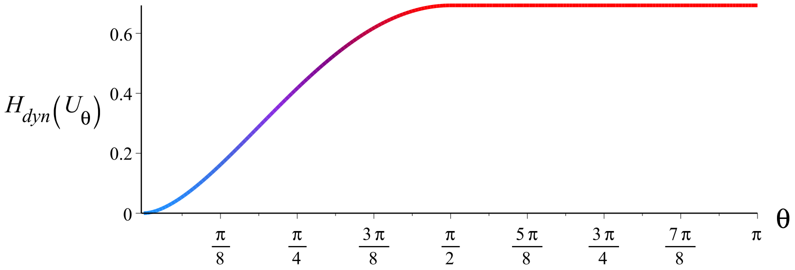
Necessary trace condition: U is chaotic $\implies$ $|\operatorname{tr} U|\leq\sqrt{d}$

Qubits

Put $U_\theta := \mathrm{diag}(1, e^{i\theta})$ for $\theta \in [0, \pi]$.

$$H^{\mathrm{dyn}}(U_\theta) = \begin{cases} \eta\left(\frac{1+\cos\theta}{2}\right) + \eta\left(\frac{1-\cos\theta}{2}\right) & \theta \le \frac{\pi}{2} \\ \ln 2 & \theta \ge \frac{\pi}{2} \end{cases}$$

$$U_\theta \text{ is chaotic} \iff \tfrac{1}{2}\pi \le \theta \le \pi \iff |\mathrm{tr}\, U_\theta| \le \sqrt{2}$$



Multiple qubits

Put $U_\theta := \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\theta})$ for $\theta \in [0, \pi]$.

$$V_{\theta,\, n} := U_\theta \otimes \underbrace{\mathbb{I}_2 \otimes \ldots \otimes \mathbb{I}_2}_{n-1}$$

Eigenvalues of $V_{\theta,\, n}$: 1 and $\mathrm{e}^{\mathrm{i}\theta}$, each with multiplicity 2^{n-1}.

$$U_\theta^{\otimes n} := \underbrace{U_\theta \otimes U_\theta \otimes \ldots \otimes U_\theta}_{n}$$

Eigenvalues of $U_\theta^{\otimes n}$: $\mathrm{e}^{m\theta\mathrm{i}}$ with multiplicity $\binom{n}{m}$, $m \in \{0, \ldots, n\}$.

$\sigma(V_{\theta,n}) = \{1, \mathrm{e}^{\mathrm{i}\theta}\}$, each with multiplicity 2^{n-1} $\Rightarrow \operatorname{tr} V_{\theta,n} = 2^{n-1}(1 + \mathrm{e}^{\mathrm{i}\theta})$
We get $|\operatorname{tr} V_{\theta,n}| \leq \sqrt{2^n} \iff \theta \in [\arccos(2^{1-n} - 1), \pi]$, so

If $V_{\theta,n}$ is chaotic, then $\theta \in [\arccos(2^{1-n} - 1), \pi]$.

We have $\arccos(2^{1-n} - 1) \to \pi$ as $n \to \infty$.

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We get $|\operatorname{tr} V_{\theta,n}| \le \sqrt{2^n} \iff \theta \in [\arccos(2^{1-n}-1), \pi]$, so

If $V_{\theta,n}$ is chaotic, then $\theta \in [\arccos(2^{1-n}-1), \pi]$.

We have $\arccos(2^{1-n}-1) \to \pi$ as $n \to \infty$.

Note that $V_{\theta,n} := U_\theta \otimes \mathbb{I}_2 \otimes \ldots \otimes \mathbb{I}_2 = U_\theta \otimes \mathbb{I}_{2^{n-1}} \sim U_\theta^{\oplus 2^{n-1}}$.

Theorem. Let $U_1 \in \mathcal{U}(\mathbb{C}^{d_1})$ and $U_2 \in \mathcal{U}(\mathbb{C}^{d_2})$. Then

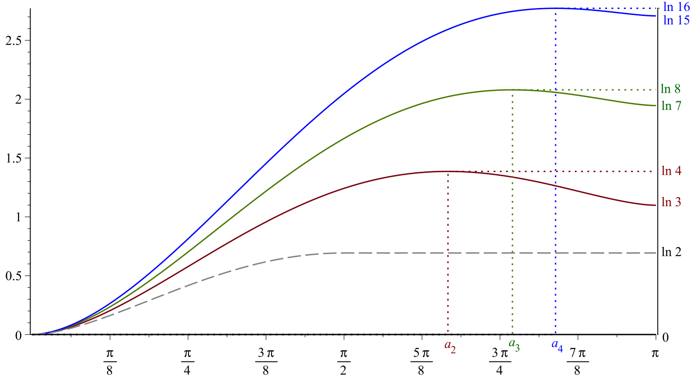
$$H^{\text{dyn}}(U_1 \oplus U_2) \ge \frac{d_1 H^{\text{dyn}}(U_1) + d_2 H^{\text{dyn}}(U_2)}{d_1 + d_2}.$$

Hence, $H^{\text{dyn}}(U_1^{\oplus n}) \ge H^{\text{dyn}}(U_1)$ for $n \in \mathbb{N}$.

$$H^{\text{dyn}}(U_\theta) \le H^{\text{dyn}}(V_{\theta,n}) \le \ln 2^n$$

$$G_n(\theta) := \eta\left(\frac{1+\cos\theta}{2}\right) + (2^n-1)\eta\left(\frac{1-\cos\theta}{2(2^n-1)}\right)$$

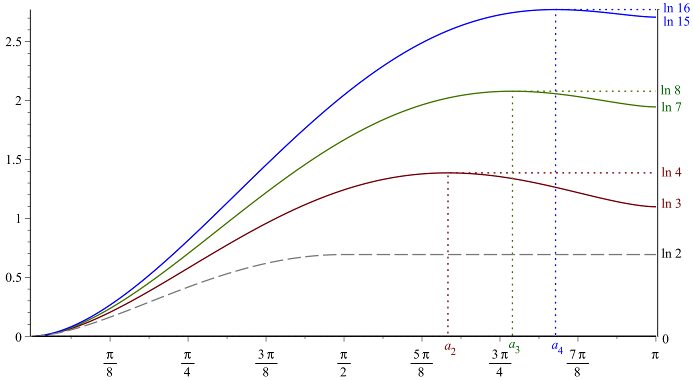
- $G_n(\theta)$ is the Shannon entropy of $\left(\frac{1+\cos\theta}{2}, \frac{1-\cos\theta}{2(2^n-1)}, \ldots, \frac{1-\cos\theta}{2(2^n-1)}\right) \in \mathbb{R}^{2^n}$
- $\max_\theta G_n(\theta) = \ln 2^n$ is attained at $\arccos(2^{1-n}-1) =: a_n$
- $G_n(\pi) = \ln(2^n-1)$



$G_m(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$

Entropy of $V_{\theta,n}$

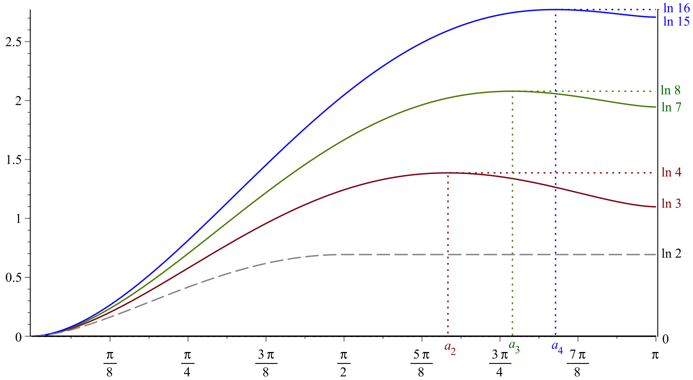
1. Theorem: $G_n(\theta) \le H^{\mathrm{dyn}}(V_{\theta,n}) \le \ln 2^n$



$G_m(\theta)$ for $m=2$, $m=3$, $m=4$ and $H^{\mathrm{dyn}}(U_\theta)$ on $\theta \in [0,\pi]$

Entropy of $V_{\theta,n}$

1. Theorem: $G_n(\theta) \leq H^{\text{dyn}}(V_{\theta,n}) \leq \ln 2^n$
2. Theorem: $H^{\text{dyn}}(V_{\theta,n}) = G_n(\theta)$ for $\theta \leq a_n$



$G_m(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$

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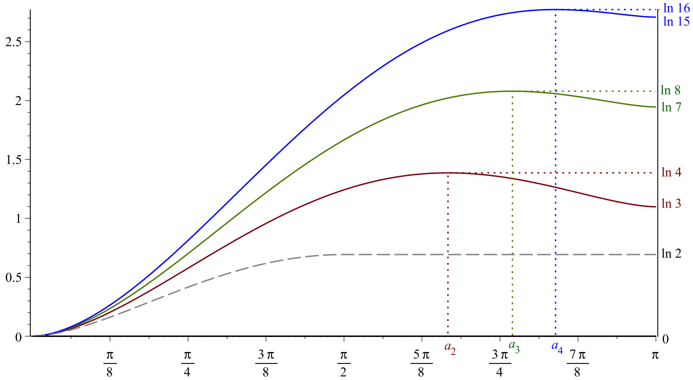
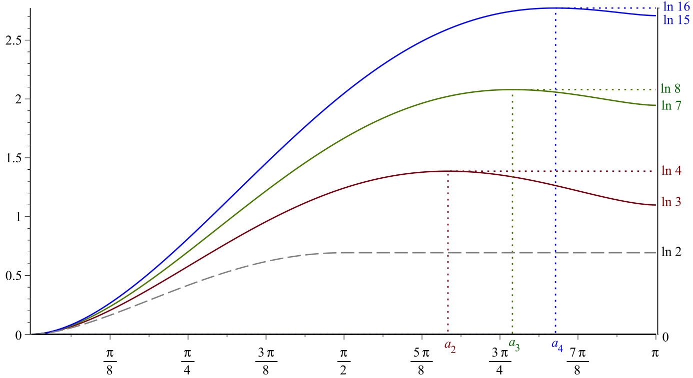


$G_m(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$

Entropy of $V_{\theta, n}$

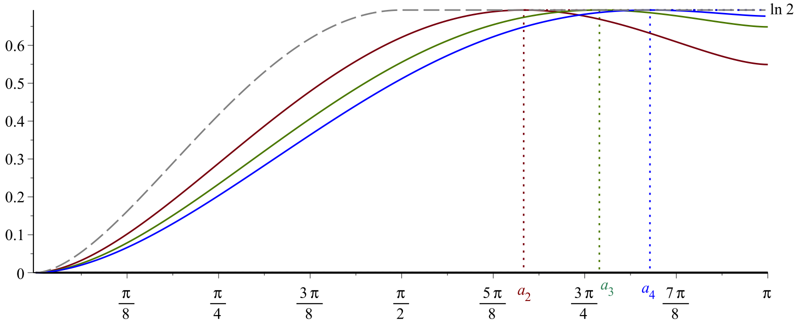
1. Theorem: $G_n(\theta) \leq H^{\text{dyn}}(V_{\theta,n}) \leq \ln 2^n$
2. Theorem: $H^{\text{dyn}}(V_{\theta,n}) = G_n(\theta)$ for $\theta \leq a_n$ and $H^{\text{dyn}}(V_{\pi,n}) = \ln 2^n$
3. Conjecture: $H^{\text{dyn}}(V_{\theta,n}) = \ln 2^n$ for $\theta > a_n$



$G_m(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$

Entropy of $V_{\theta,n}$ per qubit

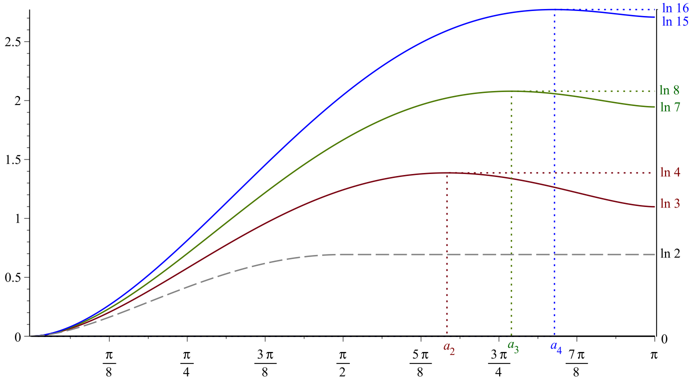
1. Theorem: $\frac{1}{n}G_n(\theta) \le \frac{1}{n}H^{\text{dyn}}(V_{\theta,n}) \le \ln 2$
2. Theorem: $\frac{1}{n}H^{\text{dyn}}(V_{\theta,n}) = \frac{1}{n}G_n(\theta)$ for $\theta \le a_n$, $\quad \frac{1}{n}H^{\text{dyn}}(V_{\pi,n}) = \ln 2$
3. Conjecture: $\frac{1}{n}H^{\text{dyn}}(V_{\theta,n}) = \ln 2$ for $\theta > a_n$



$\frac{1}{n}G_n(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0,\pi]$

Entropy of $V_{\theta,n}$

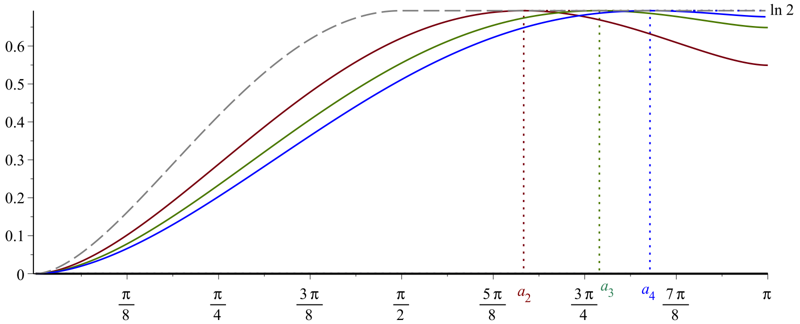
1. Corollary: $H^{\mathrm{dyn}}(U_\theta) \le H^{\mathrm{dyn}}(V_{\theta,n}) \le nH^{\mathrm{dyn}}(U_\theta)$
2. Corollary: If $1 \le m < n$ and $\theta \in (0,\pi]$, then $H^{\mathrm{dyn}}(V_{\theta,m}) < H^{\mathrm{dyn}}(V_{\theta,n})$.



$G_m(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\mathrm{dyn}}(U_\theta)$ on $\theta \in [0,\pi]$

Entropy of $V_{\theta,n}$ per qubit

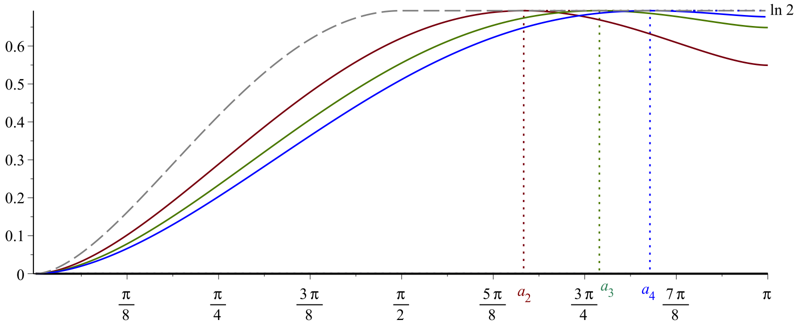
1. Thm: If $n \in \mathbb{N}$ and $\theta \in (0, a_n]$, then $\frac{1}{n+1} H^{\text{dyn}}(V_{\theta,n+1}) < \frac{1}{n} H^{\text{dyn}}(V_{\theta,n})$, so $\{\frac{1}{n} H^{\text{dyn}}(V_{\theta,n})\}_{n\in\mathbb{N}}$ is eventually strictly decreasing for $0 < \theta < \pi$.



$\frac{1}{n} G_n(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$.

Entropy of $V_{\theta,n}$ per qubit

1. Thm: If $n \in \mathbb{N}$ and $\theta \in (0, a_n]$, then $\frac{1}{n+1} H^{\text{dyn}}(V_{\theta,n+1}) < \frac{1}{n} H^{\text{dyn}}(V_{\theta,n})$, so $\{\frac{1}{n} H^{\text{dyn}}(V_{\theta,n})\}_{n\in\mathbb{N}}$ is eventually strictly decreasing for $0 < \theta < \pi$.
2. Thm: $\{\frac{1}{n} H^{\text{dyn}}(V_{\pi,n})\}_{n\in\mathbb{N}}$ is constant and equal to $\ln 2$.



$\frac{1}{n} G_n(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\text{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$.

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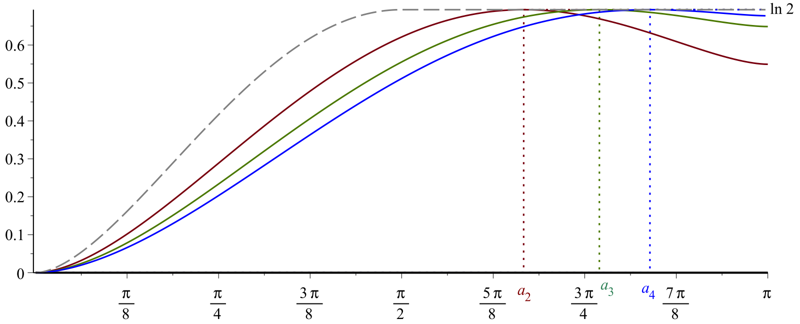
2. Thm: $\{\frac{1}{n} H^{\mathrm{dyn}}(V_{\pi,n})\}_{n\in\mathbb{N}}$ is constant and equal to $\ln 2$.
3. Conj: $\{\frac{1}{n} H^{\mathrm{dyn}}(V_{\theta,n})\}_{n\in\mathbb{N}}$ is initially (until $a_n \le \theta$) constant and equal to $\ln 2$ for $\frac{\pi}{2} < \theta < \pi$.

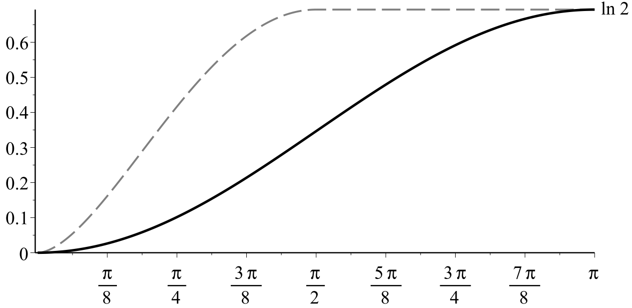


$\frac{1}{n} G_n(\theta)$ for $m = 2$, $m = 3$, $m = 4$ and $H^{\mathrm{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$.

Entropy of $V_{\theta,n}$ per qubit

$$G_n(\theta) = \eta\left(\frac{1+\cos\theta}{2}\right) + \eta\left(\frac{1-\cos\theta}{2}\right) + \frac{1-\cos\theta}{2}\ln(2^n-1)$$

$$\lim_{n\to\infty} \frac{H^{\mathrm{dyn}}(V_{\theta,n})}{n} = \frac{1-\cos\theta}{2}\ln 2$$



$\lim_{n\to\infty} \frac{1}{n} H^{\mathrm{dyn}}(V_{\theta,n})$ and $H^{\mathrm{dyn}}(U_\theta)$ for $\theta \in [0,\pi]$

We know (?) that U_π is chaotic and $U_\pi \otimes \mathbb{I}_2^{\otimes n}$ is chaotic for every $n \in \mathbb{N}$.

Def. U is **stubbornly chaotic** if $U \otimes \mathbb{I}^{\otimes n}$ is chaotic for every $n = 0, 1, 2, \ldots$

Theorem. Let $U \in \mathcal{U}(\mathbb{C}^d)$ with $\sigma(U) = \{\lambda_1, \ldots, \lambda_p\}$.

1. If U is stubbornly chaotic, then $\operatorname{tr} U = 0$.
2. If $\sigma(U)$ consists of all p-th roots of unity, each eigenvalue of U has the same multiplicity d/p, and U is chaotic, then U is stubbornly chaotic.

Examples:

- $U_\pi = \operatorname{diag}(1, -1)$ is the unique (up to phase) stubbornly chaotic unitary operator in dimension 2
- $\operatorname{diag}\left(1, \exp(\frac{2}{3}\pi \mathrm{i}), \exp(\frac{4}{3}\pi \mathrm{i})\right)$ is the unique (up to phase) stubbornly chaotic unitary operator in dimension 3.

We put $U_\theta := \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\theta})$ for $\theta \in [0, \pi]$.

$$V_{\theta,n} := U_\theta \otimes \underbrace{\mathbb{I}_2 \otimes \ldots \otimes \mathbb{I}_2}_{n-1}$$

Eigenvalues of $V_{\theta,n}$: 1 and $\mathrm{e}^{\mathrm{i}\theta}$, each with multiplicity 2^{n-1}.

$$U_\theta^{\otimes n} := \underbrace{U_\theta \otimes U_\theta \otimes \ldots \otimes U_\theta}_{n}$$

Eigenvalues of $U_\theta^{\otimes n}$: $\mathrm{e}^{m\theta\mathrm{i}}$ with multiplicity $\binom{n}{m}$, $m \in \{0, \ldots, n\}$.

We have $\sigma(U_\theta^{\otimes n}) = \sigma(V_{\theta,n})$ iff $\theta \in \{0, \pi\}$.

- $U_0^{\otimes n} = V_{0,n} = \mathbb{I}_{2^n}$,
- $U_\pi^{\otimes n}$ and $V_{\pi,n}$ are both unitarily similar to $\mathrm{diag}(\underbrace{1, \ldots, 1}_{2^{n-1}}, \underbrace{-1, \ldots, -1}_{2^{n-1}})$.

Entropy of $U_\theta^{\otimes n}$

$\operatorname{tr} U_\theta^{\otimes n} = (1 + e^{\theta i})^n \quad \Longrightarrow$

If $U_\theta^{\otimes n}$ is chaotic, then $\theta \in [\frac{\pi}{2}, \pi]$.

Theorem. If $U_1 \in \mathcal{U}(\mathbb{C}^{d_1}), U_2 \in \mathcal{U}(\mathbb{C}^{d_2})$, then

$$H^{\text{dyn}}(U_1 \otimes U_2) \geq H^{\text{dyn}}(U_1) + H^{\text{dyn}}(U_2).$$

In particular, $nH^{\text{dyn}}(U_1) \leq H^{\text{dyn}}(U_1^{\otimes n}) \leq n \ln d_1$.

Hence: if U_θ is chaotic, then $U_\theta^{\otimes n}$ is chaotic for every n, and so

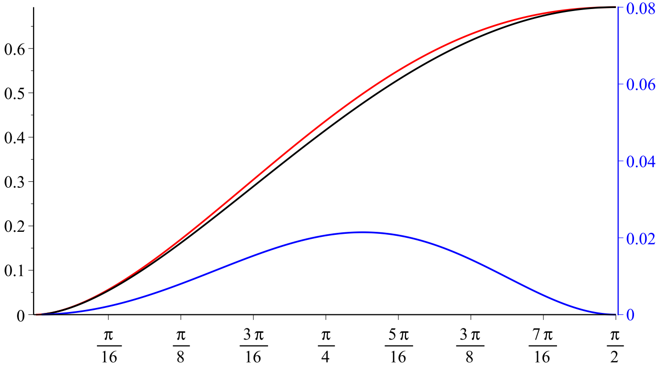
$$U_\theta^{\otimes n} \text{ is chaotic} \Leftrightarrow U_\theta \text{ is chaotic} \Leftrightarrow \theta \in [\tfrac{\pi}{2}, \pi]$$

In particular: $\operatorname{diag}(1, -1)^{\otimes n} = U_\pi^{\otimes n} \sim V_{\pi,n}$ is chaotic for every n.

$$\theta \in [0, \tfrac{\pi}{2}) \;\Rightarrow\; H^{\text{dyn}}(U_\theta) \leq \tfrac{1}{n} H^{\text{dyn}}(U_\theta^{\otimes n}) < \ln 2$$

Take V $:= P_\sigma(\frac{1}{2}F_4 \otimes \mathrm{H}_2)$, where F_4 is the Fourier matrix of size 4 and P_σ is the permutation matrix corresponding to $\sigma = (1\,4) \in S_8$.

We get $\frac{1}{3}H^{\mathrm{dyn}}(U_\theta^{\otimes 3}) > H^{\mathrm{dyn}}(U_\theta)$ for $\theta \in (0, \frac{\pi}{2})$.



$\frac{1}{3}H(U_\theta^{\otimes 3}, \mathrm{V})$ and $H^{\mathrm{dyn}}(U_\theta)$ on $\theta \in [0, \pi]$

against the blue axis: $\frac{1}{3}H(U_\theta^{\otimes 3}, \mathrm{V}) - H^{\mathrm{dyn}}(U_\theta)$

Entropy of $U_\theta^{\otimes n}$

The sequence $(H^{\mathrm{dyn}}(U_\theta^{\otimes n}))_{n\in\mathbb{N}}$ is superadditive, i.e., we have

$$H^{\mathrm{dyn}}(U_\theta^{\otimes n_1+n_2}) \geq H^{\mathrm{dyn}}(U_\theta^{\otimes n_1}) + H^{\mathrm{dyn}}(U_\theta^{\otimes n_2}) \quad \text{for } n_1, n_2 \in \mathbb{N}.$$

It follows that

$$\lim_{n\to\infty} \frac{1}{n} H^{\mathrm{dyn}}(U_\theta^{\otimes n}) = \sup_{n\in\mathbb{N}} \frac{1}{n} H^{\mathrm{dyn}}(U_\theta^{\otimes n}).$$

$$\lim_{n\to\infty} \frac{1}{n} H^{\mathrm{dyn}}(U_\theta^{\otimes n}) = \begin{cases} \ln 2 & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \\ 0 & \text{if } \theta = 0 \end{cases}$$

$$H^{\mathrm{dyn}}(U_\theta) < \lim_{n\to\infty} \frac{1}{n} H^{\mathrm{dyn}}(U_\theta^{\otimes n}) < \ln 2 \qquad \text{if } 0 < \theta < \frac{\pi}{2}$$