MULTIPARTITE STEERING INEQUALITIES BASED ON ENTROPIC UNCERTAINTY RELATIONS

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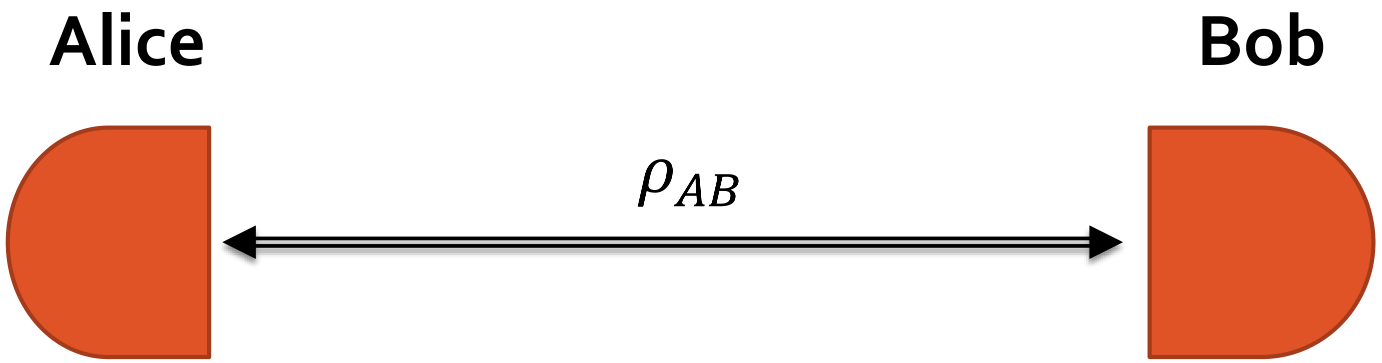
Outline

- **Quantum correlations**
- **Entropic uncertainty relations**
- **Steering criteria based on entropic uncertainty relations**
- **Perspectives**

A. Riccardi, C. Macchiavello and L. Maccone, Multipartite steering inequalities based on entropic uncertainty relations, **Phys. Rev. A 97, 052307 (2018)**

Bipartite Scenario



$X_A = \sum_a x_a \Pi_a^A$ $\quad\quad$ $Z_B = \sum_b z_b \Pi_b^B$

Figure of merit for correlations

$$p(x_a, z_b) = Tr[\rho_{AB} \Pi_a^A \otimes \Pi_b^B]$$

QM allows correlations between measurement outcomes that cannot be reproduced classically

States

Single system A

Pure state

$$|\psi\rangle \in H_A \quad \text{such that} \quad |\langle\psi|\psi\rangle|^2 = 1$$

Mixed state

$$\rho_A = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

$$\sum_i p_i = 1$$

Bipartite system A-B

Pure state:

$$|\psi\rangle \in H_A \otimes H_B \quad \text{such that} \quad |\langle\psi|\psi\rangle|^2 = 1$$

Product state $\quad |\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$

Entangled state $\quad |\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$

The states of the subsystems are mixed!

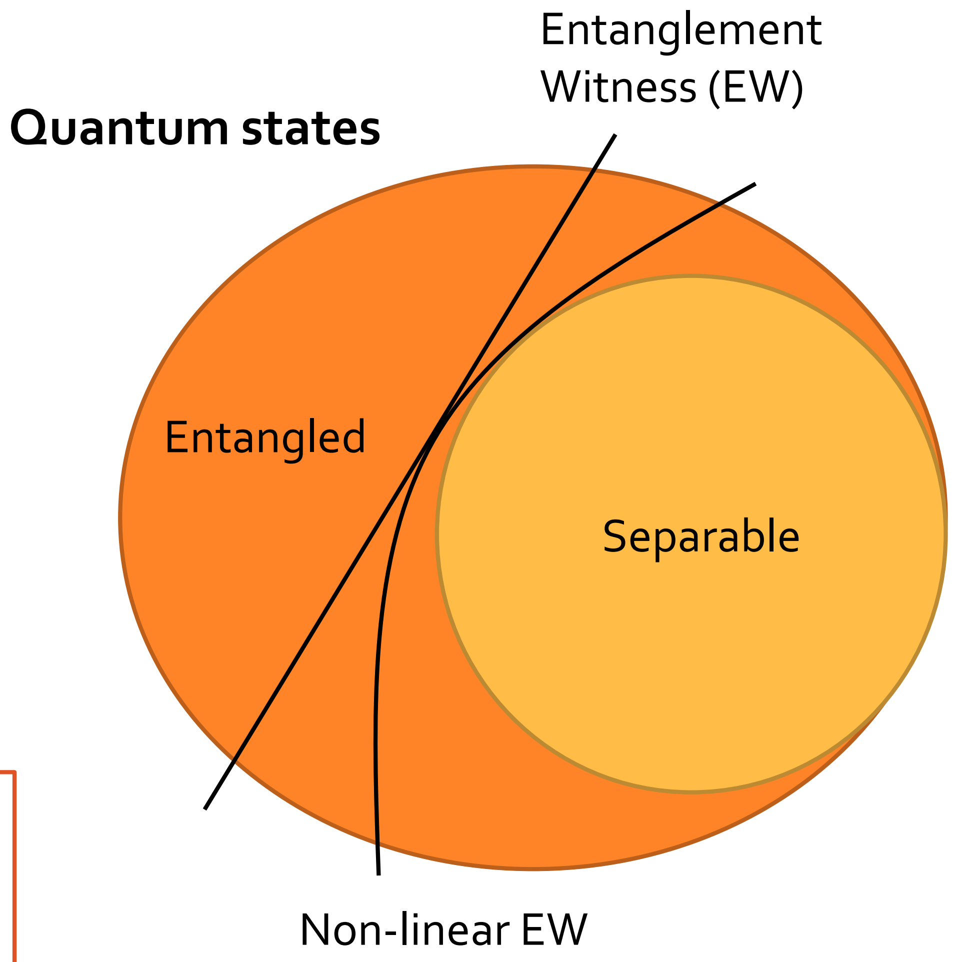
Bipartite mixed states

$$\rho_{AB} = \sum_i p_i \, |\psi_i\rangle_{AB} \; {}_{AB}\langle\psi_i|$$

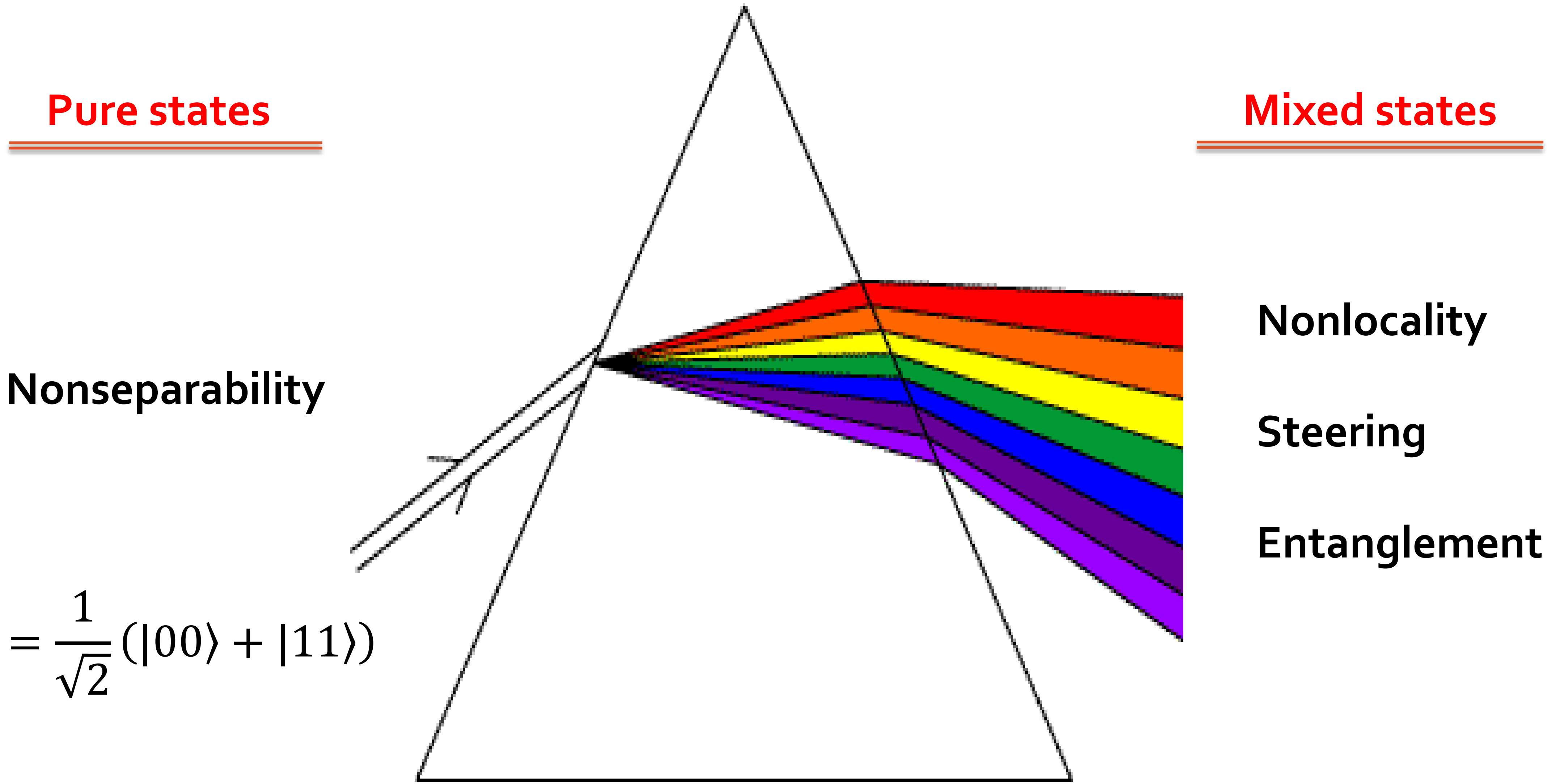
Separable states: $\rho_{AB} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i$

Entangled state: $\rho_{AB} \neq \sum_i p_i \, \rho_A^i \otimes \rho_B^i$

Any state that cannot be written as a convex combination of product states is **entangled**



Quantum correlations



$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Hierarchy of Quantum correlations

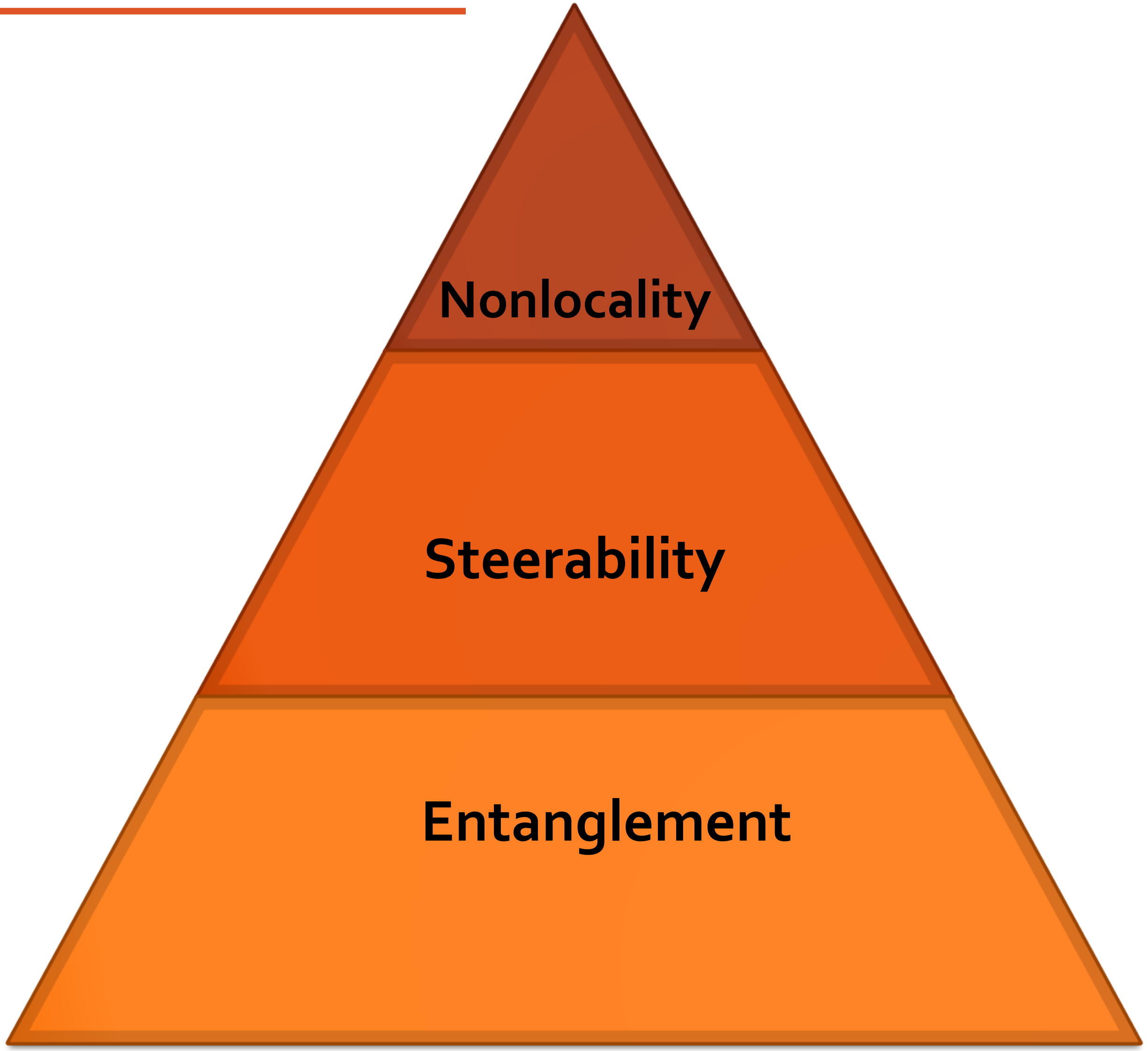
Figure of merit

$$p(x_a, z_b) = Tr[\rho_{AB}\Pi_a^A \otimes \Pi_b^B]$$

For studying the nature of the different types of correlations:

1. We define a **model** for each type (e.g the separability model) and we analyze how we can write $p(a_j, b_k)$ within that model.

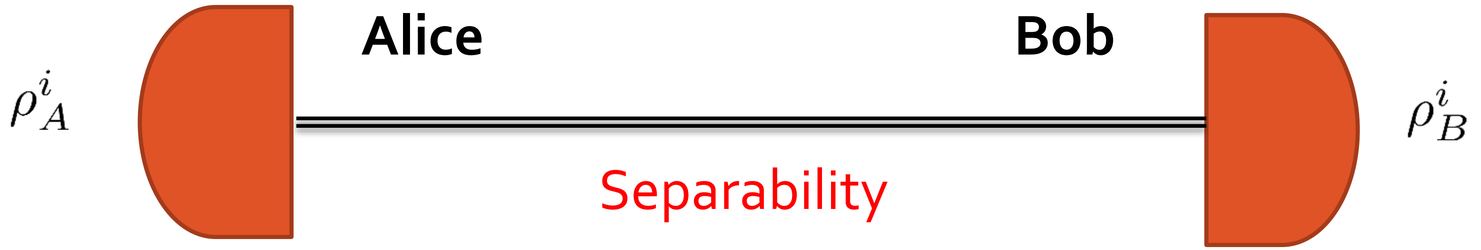
2. Whenever that model does not hold then the corresponding type of correlation exist (e.g entanglement)



N. Brunner, et al. Rev. Mod. Phys. 86.2, 419 (2014). // R. F. Werner, Phys. Rev. A 40, 4277 (1989). // G. Adesso et. al, J. Phys. A 49 (47), 473001 (2016).

Separability model

$$\rho_{AB} = \sum_i p_i\, \rho_A^i \otimes \rho_B^i$$



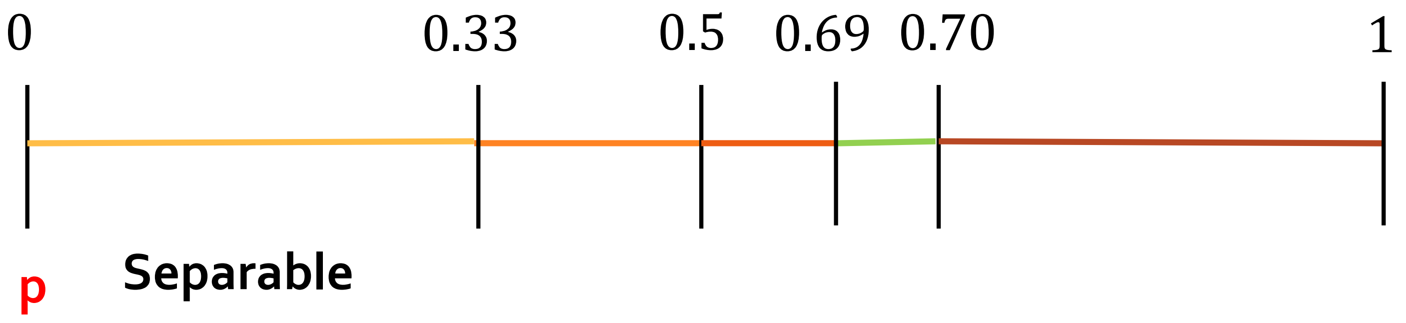
We say that **the correlations demonstrate entanglement** iff for every possible choice of A and B the joint measurement probabilities cannot be written as:

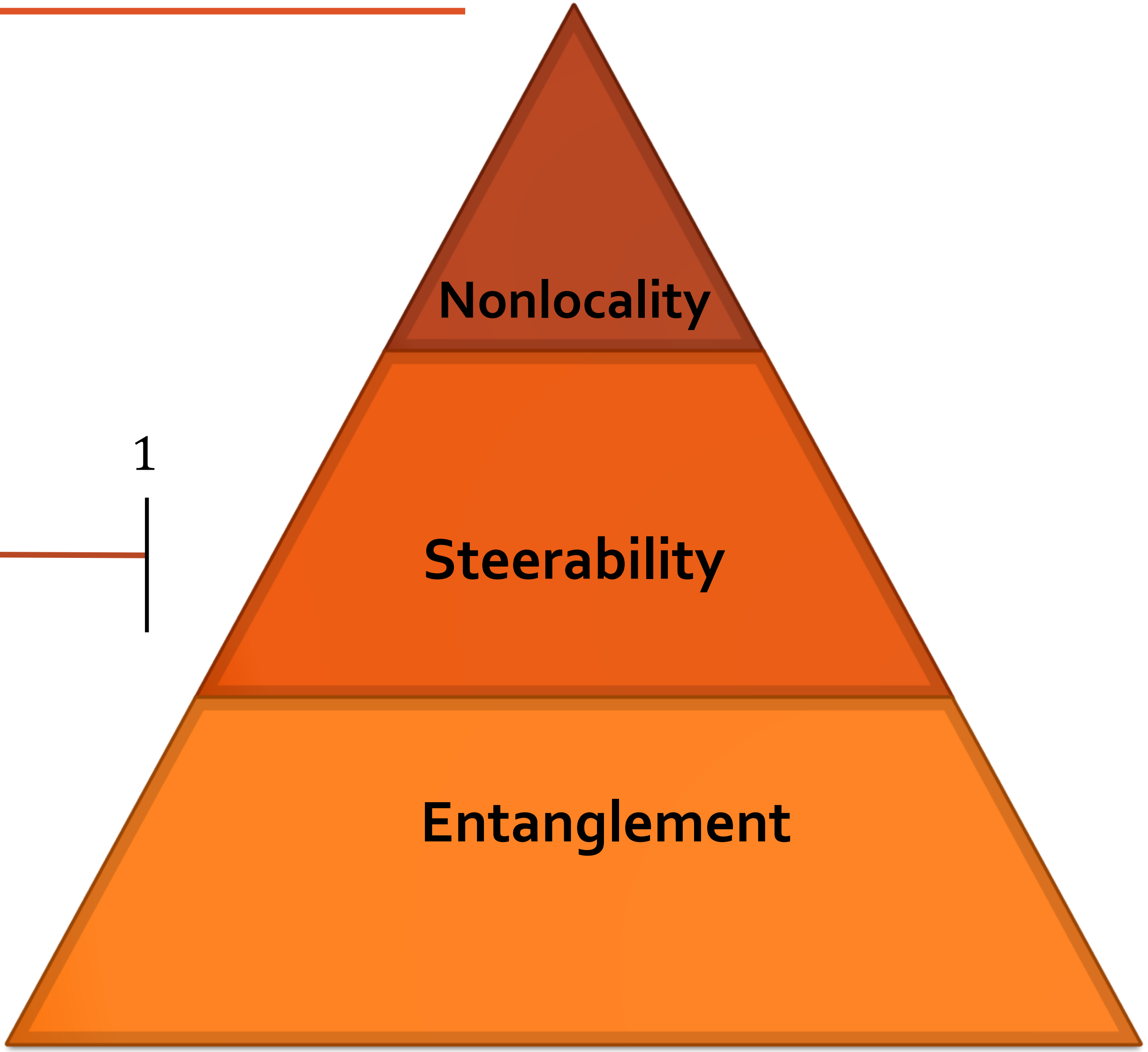
$$p(a_j, b_k) \neq \sum_i p_i p_Q(a_j|i) p_Q(b_k|i)$$

$$\rho_Q\left(a_j|i\right) = \mathrm{Tr}\left[\rho_A^i \Pi_j^A\right]$$

Hierarchy of Quantum correlations

$$\rho_{Werner} = p|\phi^+\rangle\langle\phi^+| + \frac{p}{4} I \otimes I$$





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Hierarchy of Quantum correlations

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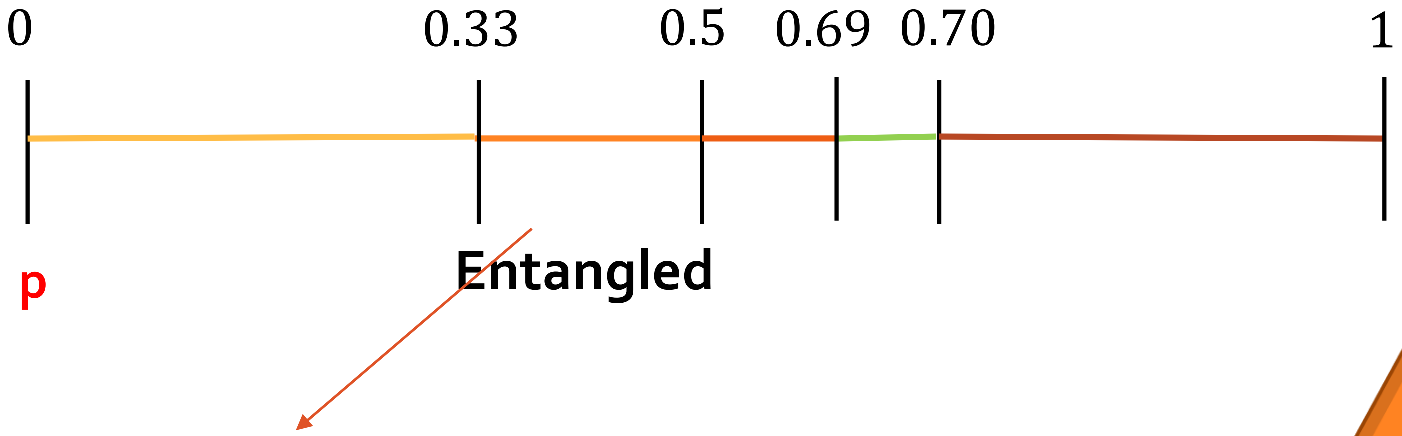
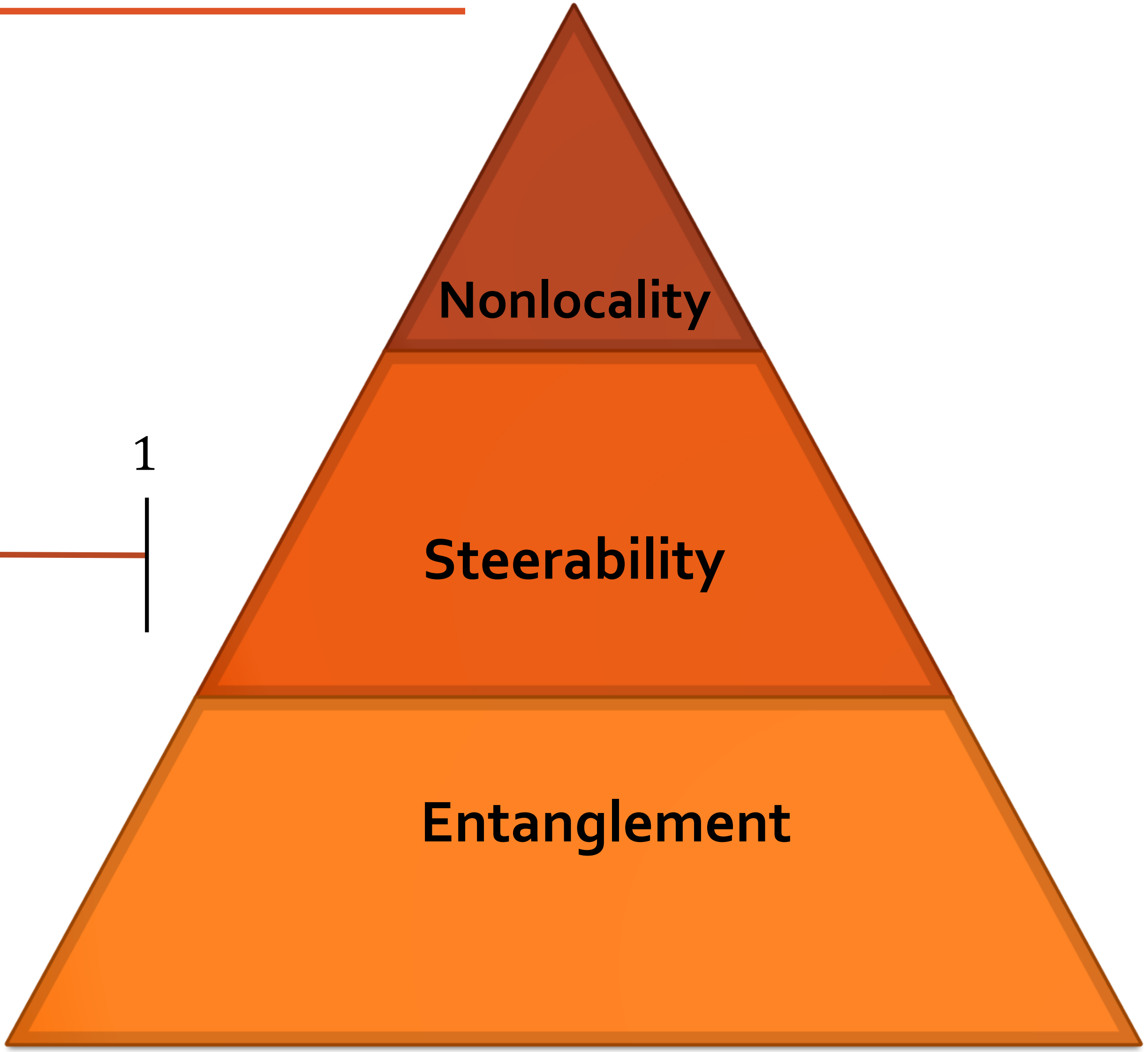


Entangled states cannot be prepared with LOCC (Local operations and classical communications)



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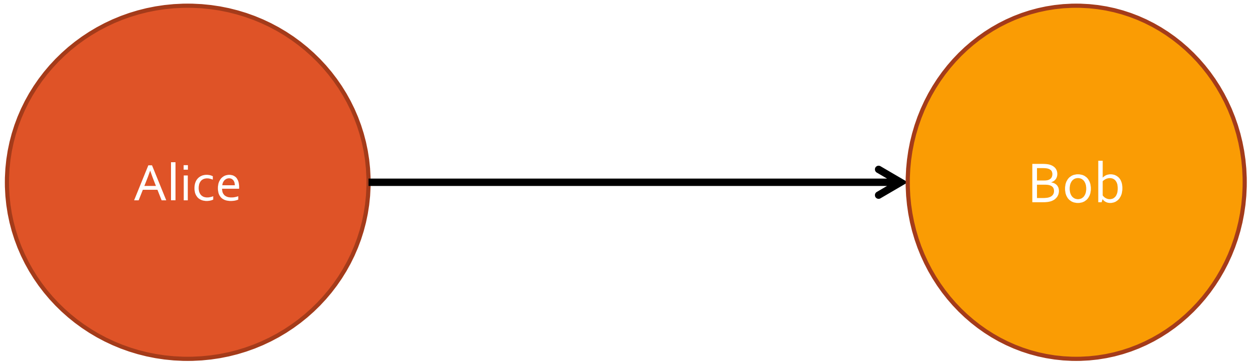
Quantum steering

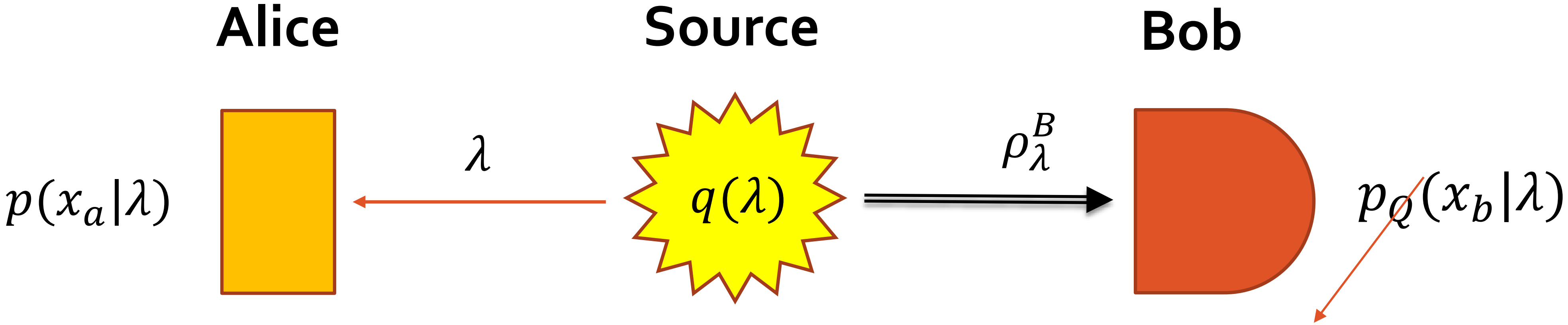
Quantum steering is a type of quantum correlation which enables one part to influence the state of the others, with which it shares an entangled state, by applying local measurements.

To have quantum steering we must require that the **correlations** encoded in the state of system **cannot be reproduced with an underlying local hidden state model**

Steering scenario



Local hidden state model



$$p_Q(x_b|\lambda) = Tr[\rho_\lambda^B \Pi_b^B]$$

$$p(x_a, x_b) = \sum_\lambda q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda)$$

LHS

Bipartite steering

We say that **the correlations demonstrate quantum steering** iff:

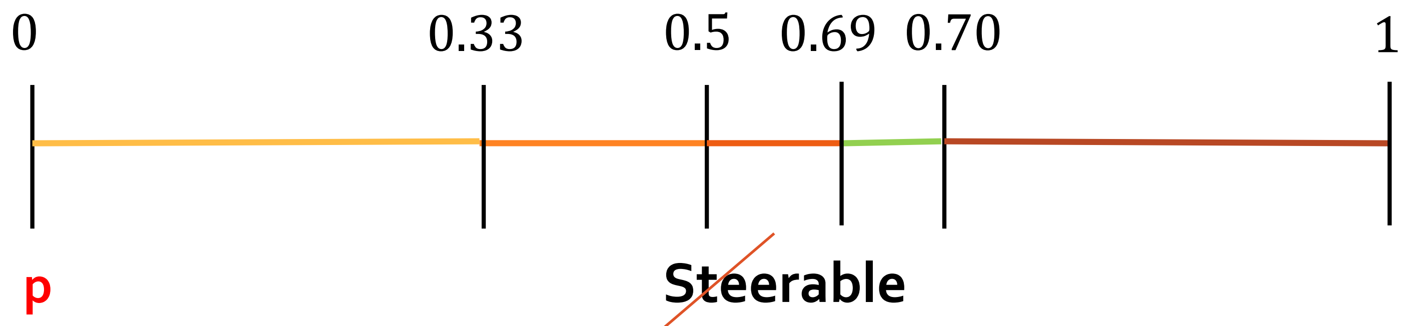
For every possible measurements X_A and X_B the joint measurement probabilities cannot be written as:

$$p(x_a, x_b) \neq \sum_{\lambda} q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda)$$

$p_Q(x_b|\lambda)$ is originating from pre-determined state ρ_λ^B, that depends on the classical variable λ. Bob can explain its results in terms of local states.

Hierarchy of Quantum correlations

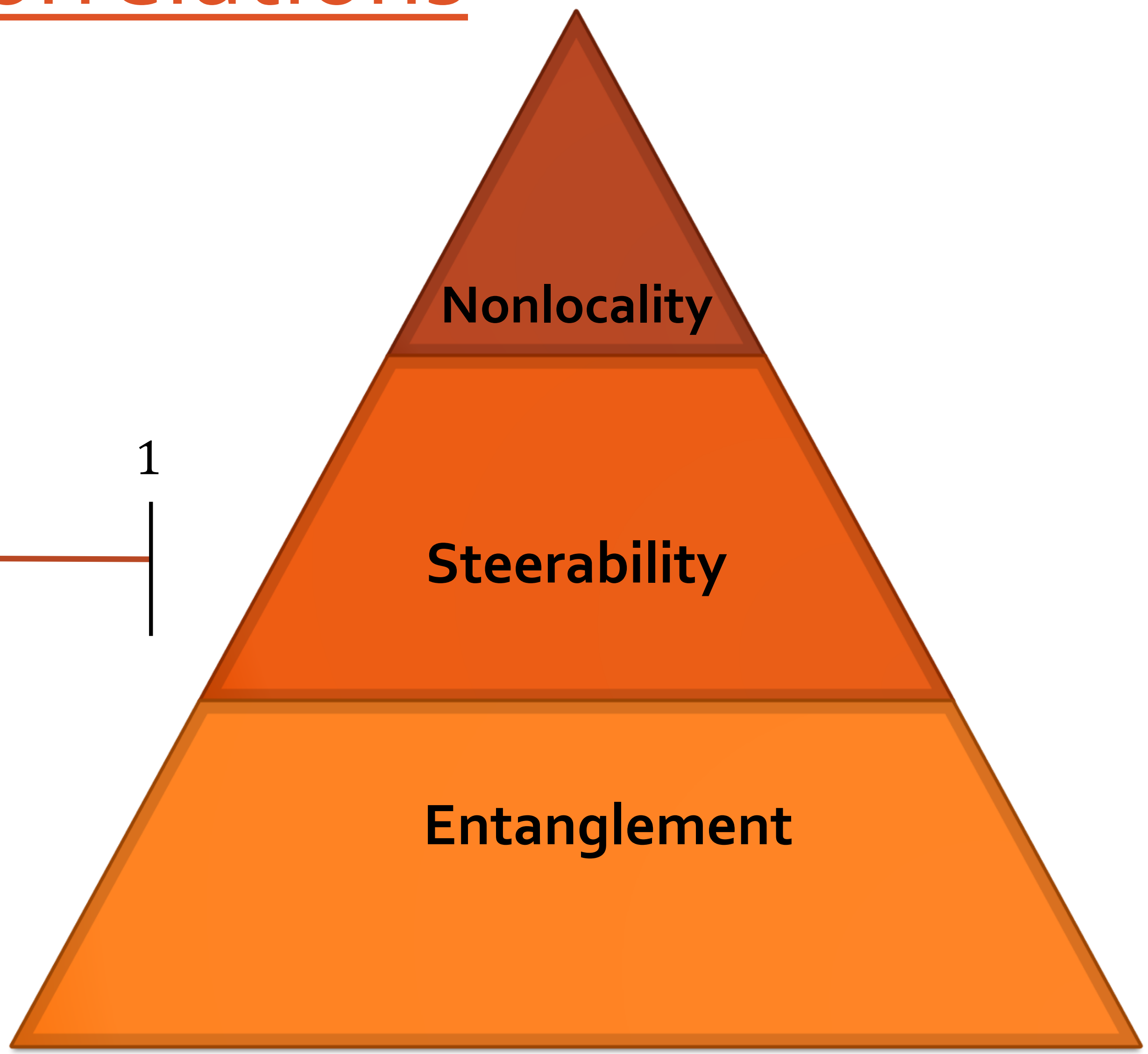
$$\rho_{Werner} = p|\phi^+\rangle\langle\phi^+| + \frac{p}{4} I \otimes I$$



Steerable

No local Hidden state models

Violation of a steering criterion



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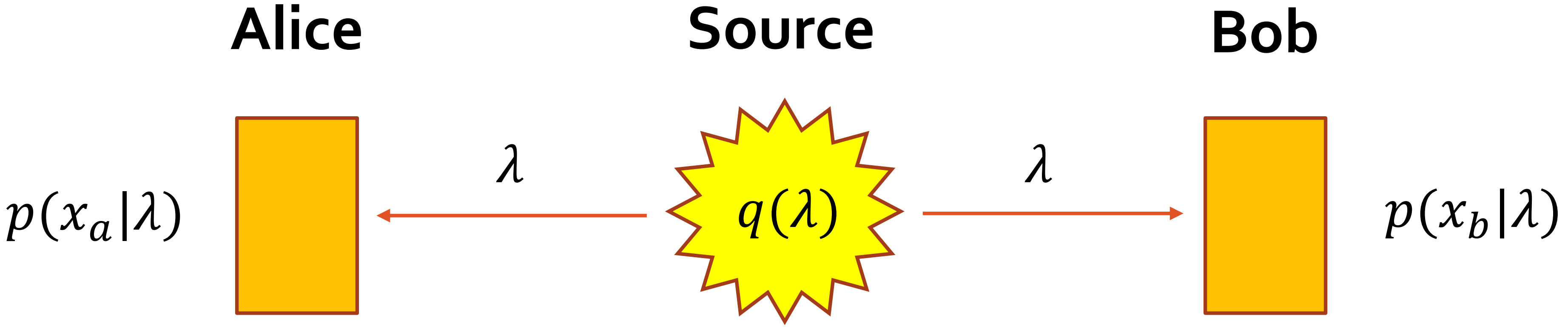
Local hidden state model



$$p(x_a, x_b) = \sum_{\lambda} q(\lambda) p(x_a|\lambda) p(x_b|\lambda)$$

LHV

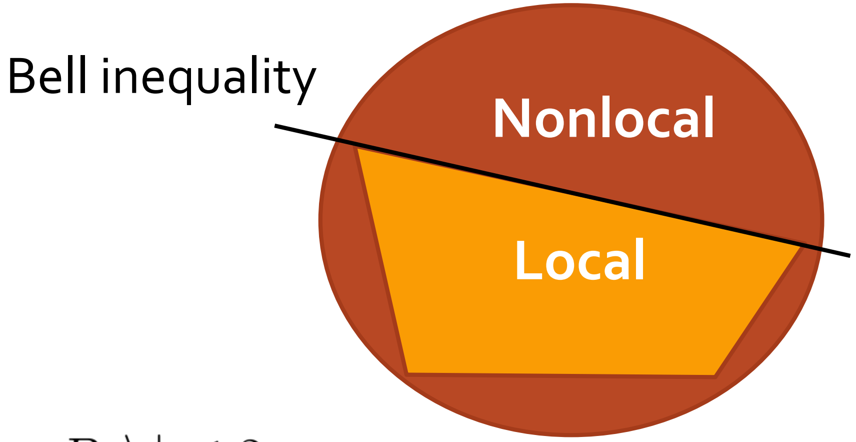
Nonlocality

We say that **the correlations demonstrate nonlocality** iff for every possible choice of A and B the joint measurement probabilities cannot be written in terms of a local hidden variable (LHV) model.

$$p(x_a, x_b) \neq \sum_{\lambda} q(\lambda) p(x_a|\lambda) p(x_b|\lambda)$$

Bell inequalities: relations that are valid under the assumption of locality.

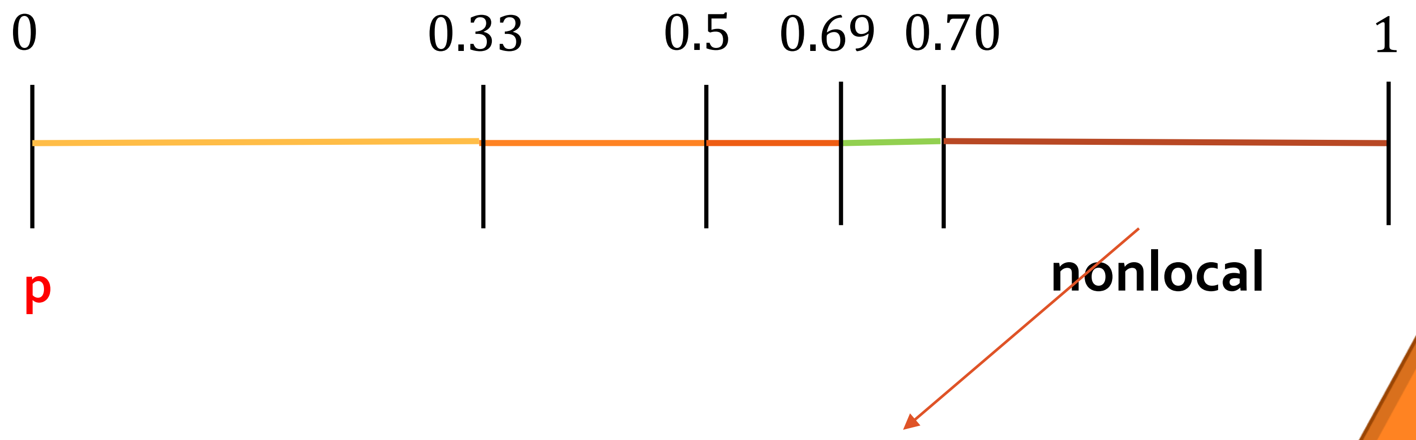
If a state violates a Bell inequality then it must be nonlocal.



CHSH inequality $S = |\langle A_0 \otimes B_0 \rangle + \langle A_1 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle - \langle A_1 \otimes B_1 \rangle| \leq 2$

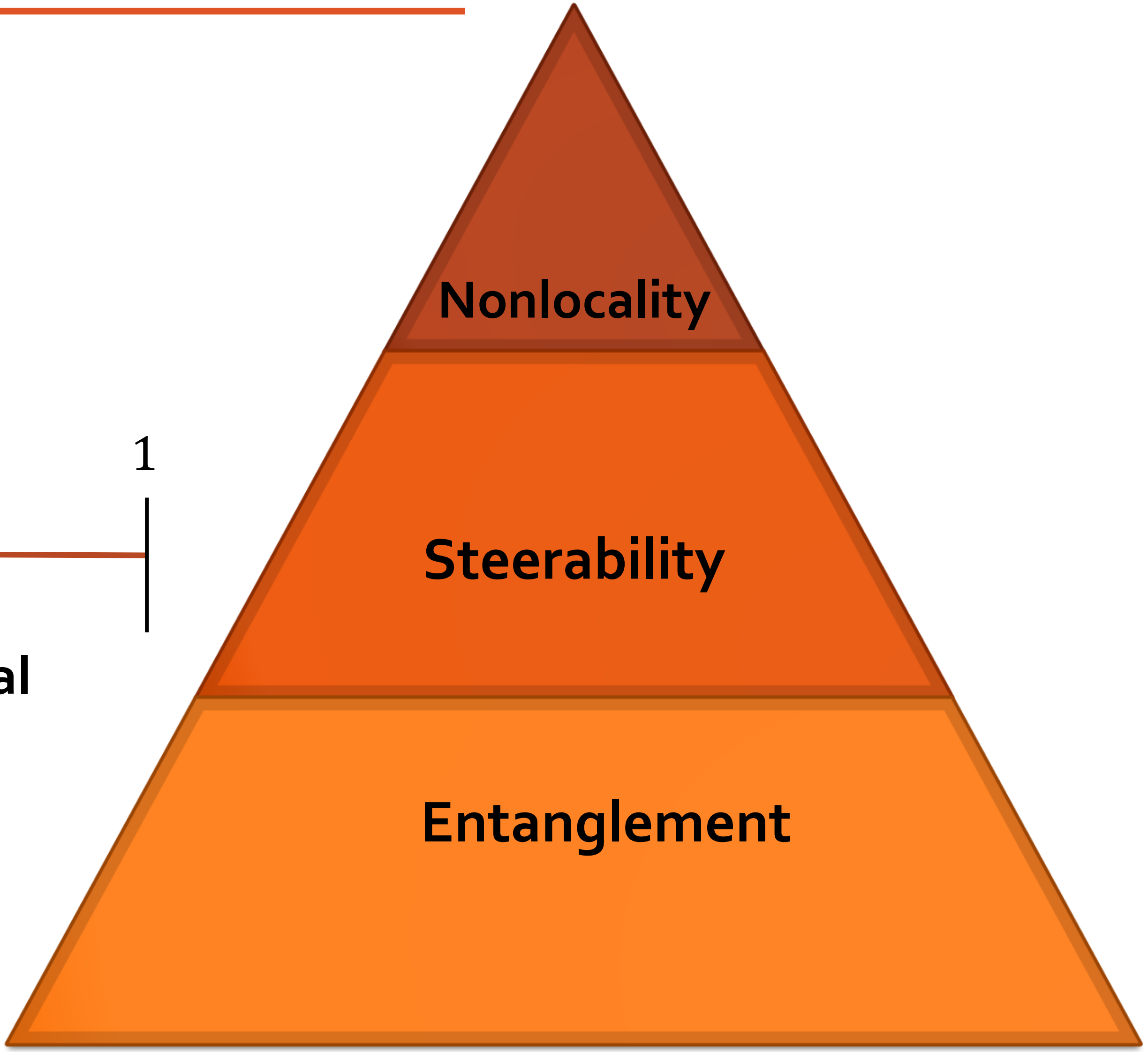
Hierarchy of Quantum correlations

$$\rho_{Werner} = p|\phi^+\rangle\langle\phi^+| + \frac{p}{4} I \otimes I$$



No hidden variable models

Violation of a Bell inequality



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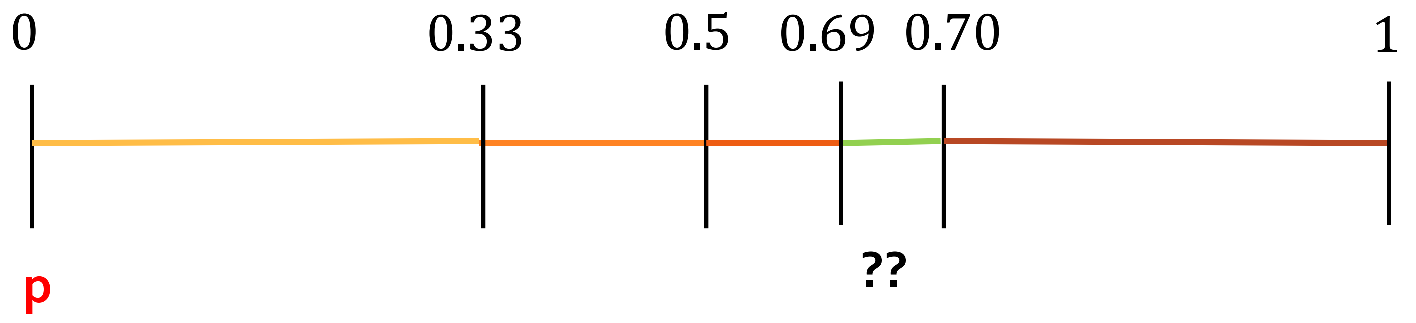
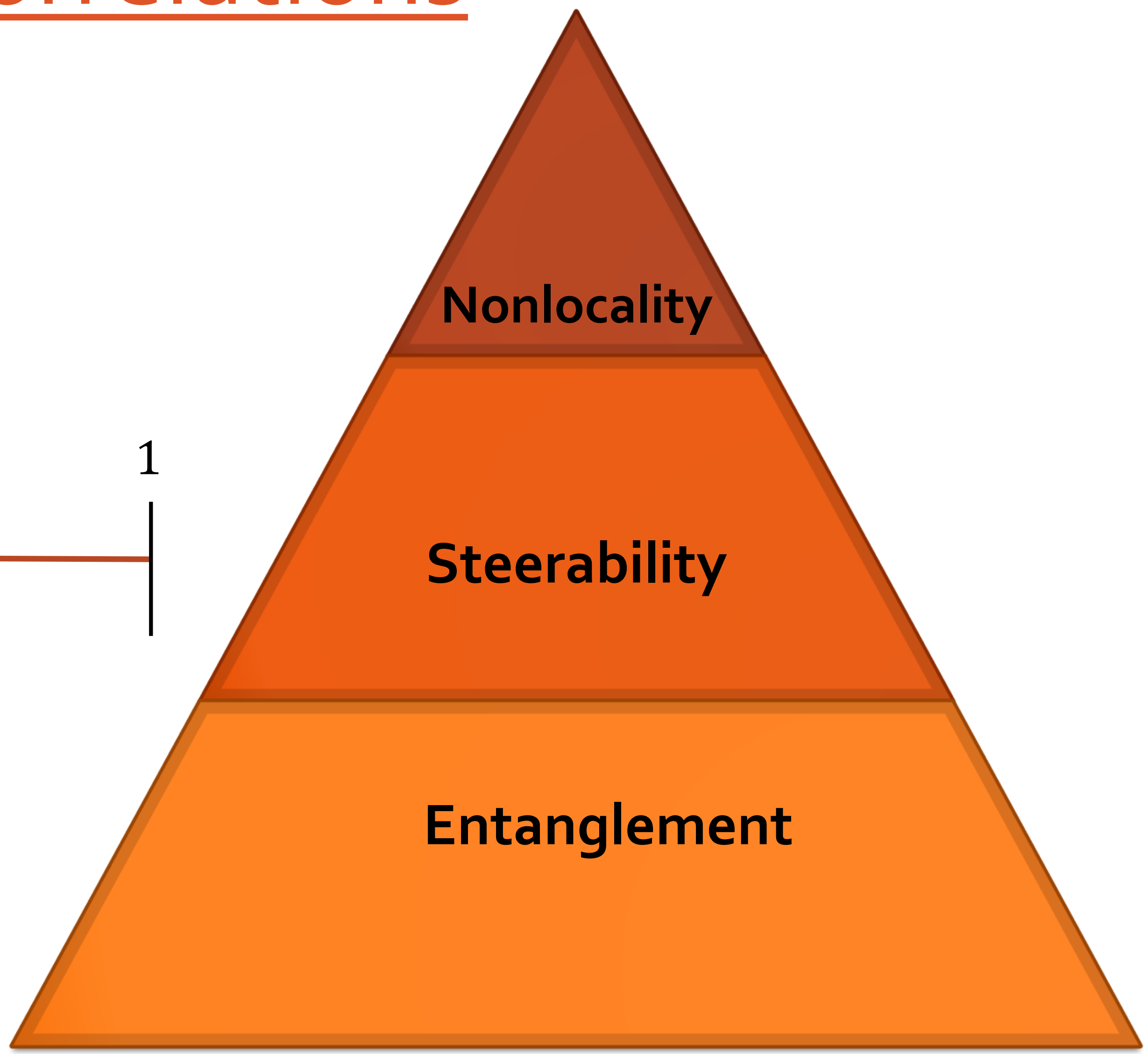
Hierarchy of Quantum correlations

$$\rho_{Werner} = p|\phi^+\rangle\langle\phi^+| + \frac{p}{4} I \otimes I$$





In this region it is not know if Werner states are steerable or nonlocal

N. Brunner, et al. Rev. Mod. Phys. 86.2, 419 (2014). // R. F. Werner, Phys. Rev. A 40, 4277 (1989).

Tripartite Scenario

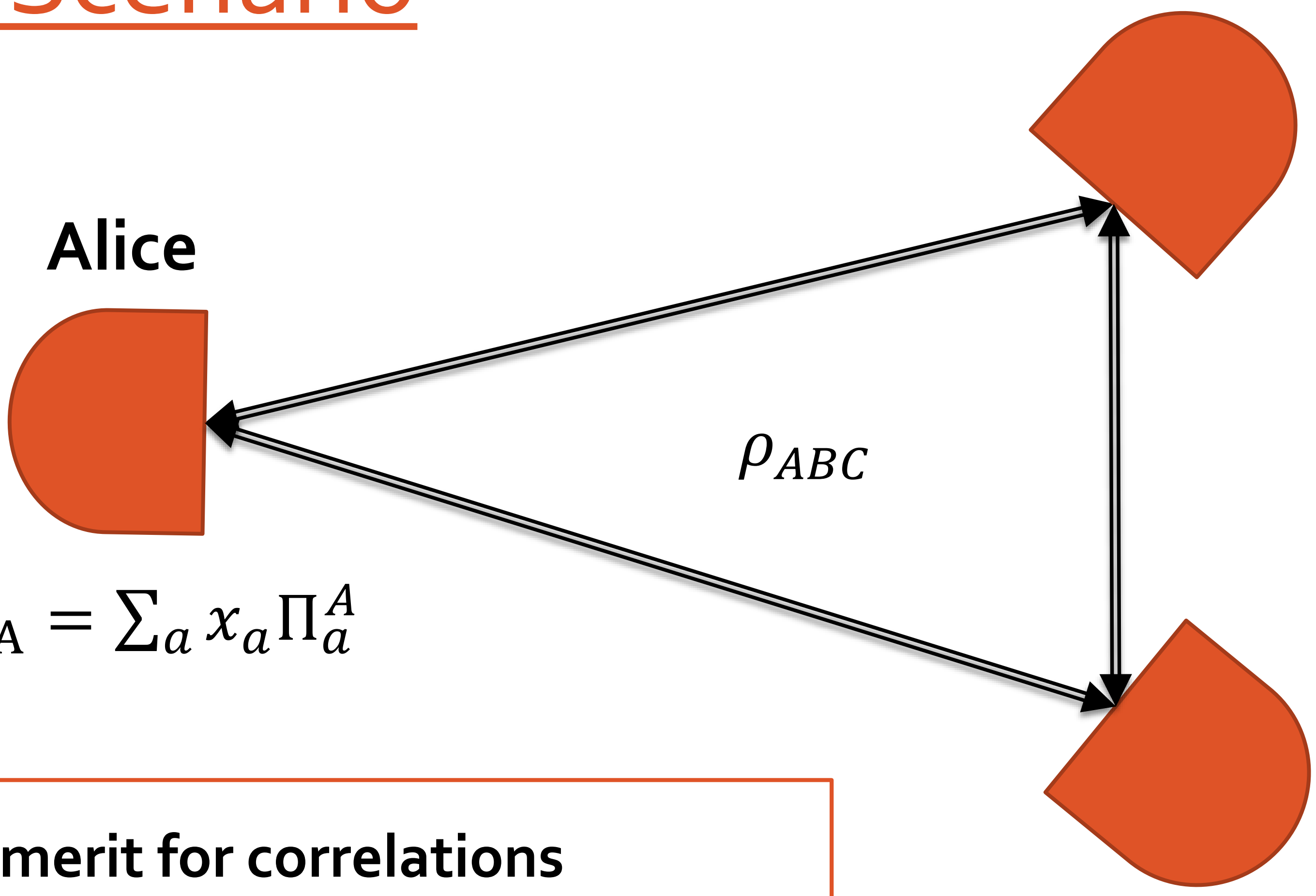


Figure of merit for correlations

$$p(x_a, z_b, y_c) = Tr[\rho_{ABC} \Pi_a^A \otimes \Pi_b^B \otimes \Pi_c^C]$$

Multipartite quantum steering

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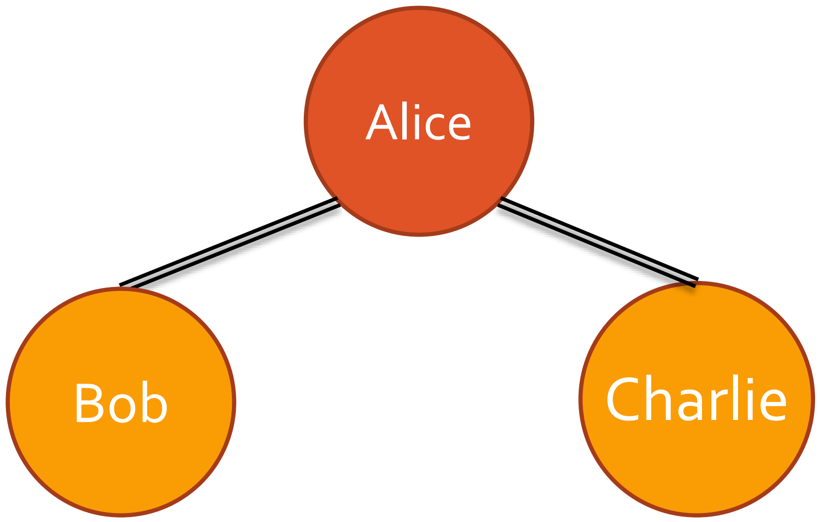
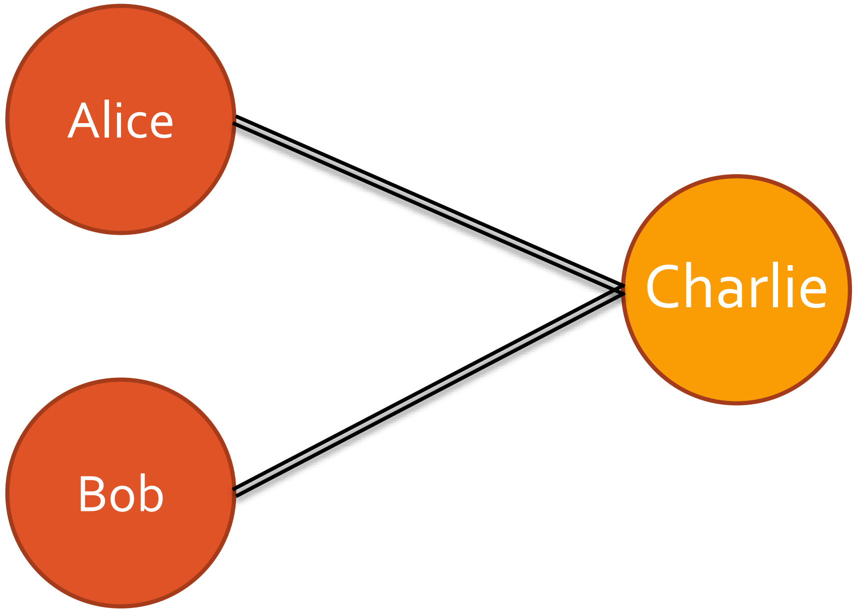
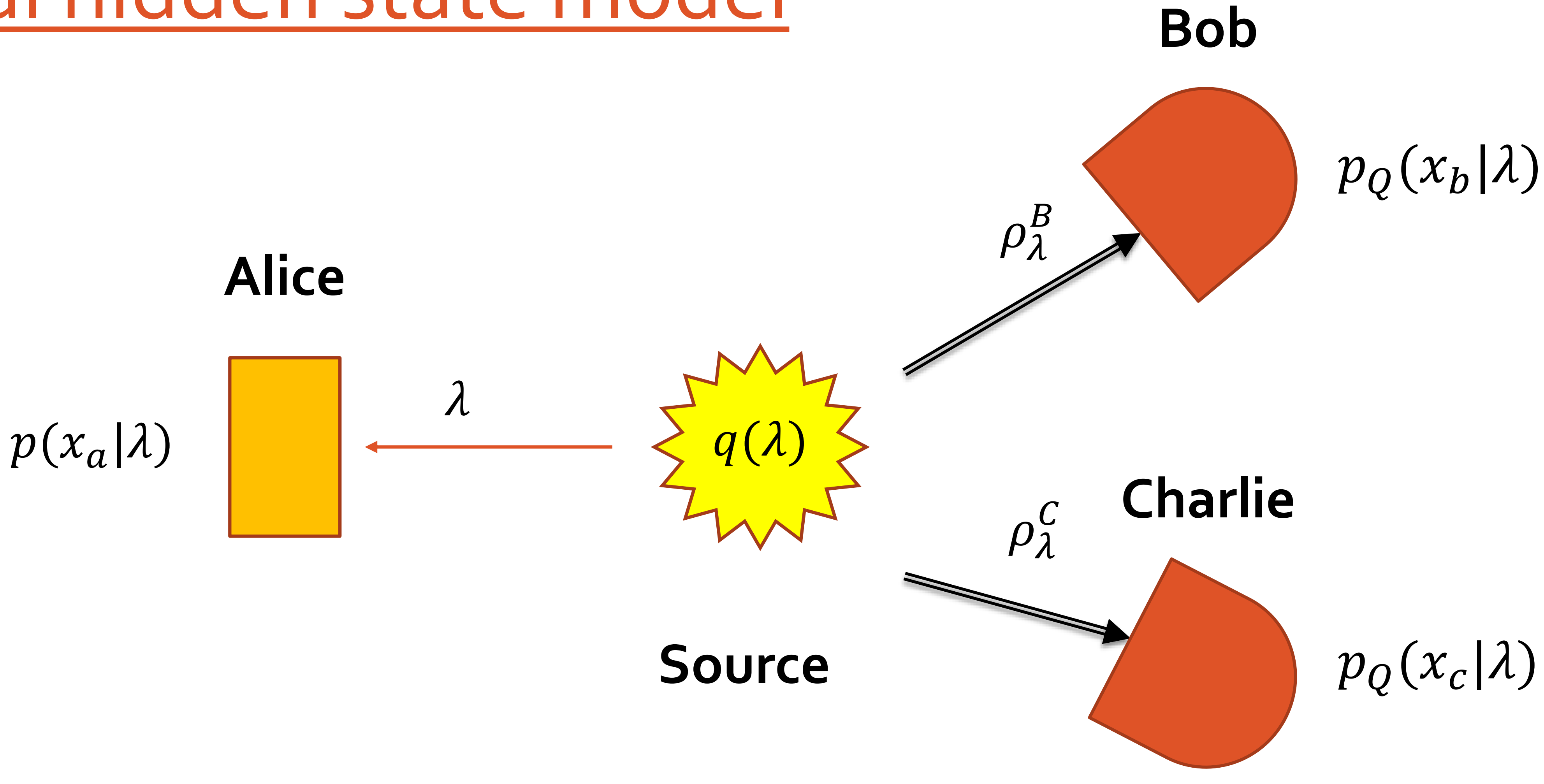
One-to-two steering
Alice wants to steer both Bob and Charlie.



Two-to-one steering
Alice and Bob want to steer Charlie.



Local hidden state model



$$p(x_a, x_b, x_c) = \sum_\lambda q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda) p_Q(x_c|\lambda)$$

One-to-two scenario: simple tripartite steering

We say that **the correlations demonstrate tripartite quantum steering** iff:

(1) For every possible measurements X_A, X_B and X_C the joint measurement probabilities cannot be written as:

$$p(x_a, x_b, x_c) \neq \sum_{\lambda} q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda) p_Q(x_c|\lambda)$$

Local Hidden State Model

$p_Q(x_b|\lambda)$ and $p_Q(x_c|\lambda)$ are computed over the pre-determined state $\rho_\lambda^B \otimes \rho_\lambda^C$, for a given classical variable λ.

(2) Entanglement between Alice and Bob+Charlie is detected.

D. Cavalcanti et. al, Nat. Commun. 6, 7941 (2015).

One-to-two scenario: genuine tripartite steering

We say that **the correlations demonstrate genuine tripartite quantum steering** iff for every possible measurements X_A, X_B and X_C the joint measurement probabilities cannot be written as:

$$p(x_a, x_b, x_c) \neq \sum_{\nu} q_A(\nu) p(x_a|\lambda) p_Q(x_b, x_c|\lambda) + \sum_{\gamma} q_B(\gamma) p(x_a|\gamma) p(x_c|\gamma, x_a) p_Q(x_b|\gamma) + \\ + \sum_{\omega} q_C(\omega) p(x_a|\omega) p(x_b|\omega, x_a) p_Q(x_c|\omega)$$

Genuine multipartite steering is the analogous of genuine multipartite entanglement for this type of quantum correlation.

D. Cavalcanti et. al, Nat. Commun. 6, 7941 (2015).

One-to-two scenario: genuine tripartite steering

Alice cannot steer Bob and Charlie.

$$p(x_a, x_b, x_c) \neq \sum_{\nu} q_A(\nu) p(x_a|\lambda) p_Q(x_b, x_c|\lambda) + \sum_{\gamma} q_B(\gamma) p(x_a|\gamma) p(x_c|\gamma, x_a) p_Q(x_b|\gamma) +$$

$$+ \sum_{\omega} q_C(\omega) p(x_a|\omega) p(x_b|\omega, x_a) p_Q(x_c|\omega)$$

One-to-two scenario: genuine tripartite steering

Alice can steer only Charlie.

$$p(x_a, x_b, x_c) \neq \sum_{\nu} q_A(\nu) p(x_a|\lambda) p_Q(x_b, x_c|\lambda) + \sum_{\gamma} q_B(\gamma) p(x_a|\gamma) p(x_c|\gamma, x_a) p_Q(x_b|\gamma) + \\ + \sum_{\omega} q_C(\omega) p(x_a|\omega) p(x_b|\omega, x_a) p_Q(x_c|\omega)$$

One-to-two scenario: genuine tripartite steering

$$p(x_a, x_b, x_c) \neq \sum_{\nu} q_A(\nu) p(x_a|\lambda) p_Q(x_b, x_c|\lambda) + \sum_{\gamma} q_B(\gamma) p(x_a|\gamma) p(x_c|\gamma, x_a) p_Q(x_b|\gamma) + \sum_{\omega} q_C(\omega) p(x_a|\omega) p(x_b|\omega, x_a) p_Q(x_c|\omega)$$

Alice can steer only Bob.

Uncertainty relations (UR)

Preparation uncertainty relations: not all properties of a quantum system can be exactly defined at once.

.We cannot prepare states that are both eigenstates of X and P

Quantum complementarity: there exist complementary properties which can be assigned to a system, but that cannot have joint definite values.

If we know a value of a property, then all the values of the complementary property are completely unknown.

Even if the UR originated from complementarity, they allow to balance the effects of complementarity. Indeed **UR tell us that complementary properties can be defined at least partially, as long as we do not require them to be determined with perfect precision.**

Entropic uncertainty relations (EUR)

Heisenberg-Robertson's UR: $$\Delta^2 X \Delta^2 Z \geq \frac{1}{4} |\langle [X, Z] \rangle|^2$$

Entropic uncertainty relations are inequalities that express the **uncertainty** relations as **lower bounds of the sum of Shannon entropies** of probability distribution of measurement outcomes.

Maaseen and Uffink (1988): $$\boxed{H(X) + H(Z) \geq -\log_2 \alpha_{XZ}} \qquad \alpha_{XZ} = \max_{j,k} \left| \langle x_k | z_j \rangle \right|^2$$

State-indepedendent bound

We can define uncertainty relations for more than two observables

H. Maassen, J. B. M. Uffink Phys. Rev. Lett. **60**, 1103 (1988) // A. R., C. Macchiavello, L Maccone Phys. Rev. A 95 (3), 032109 (2017)

Entropic uncertainty relations (EUR)

Complementary observables (MUBs) $|\langle x_k | z_j \rangle| = \frac{1}{\sqrt{d}}$ ⟷ $H(X) + H(Z) \geq \log_2 d$

EUR for three complementary observables $H(X) + H(Z) + H(Y) \geq \frac{3}{2} \log_2 d$

Can we use EUR to study quantum correlations encoded in a state?

$$H(X_B, X_C | X_A) + H(Z_B, Z_C | Z_A) \geq ?$$

L. Maccone et. al, Phys. Rev. Lett. **114**, 130401 (2015) // J. Schneeloch et al. Phys. Rev. A 87, 062103 (2013).

Simple steering inequality

(1) Assume that the state is nonsteerable from Alice to Bob and Charlie:

$$p(x_a, x_b, x_c) = \sum_\lambda q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda) p_Q(x_c|\lambda)$$

(2)

$$H(X_B, X_C|X_A) + H(Z_B, Z_C|Z_A) \geq \sum_\lambda q(\lambda)[H_\lambda(X_B, X_C) + H_\lambda(Z_B, Z_C)]$$

$$\geq -\sum_\lambda q(\lambda) \log_2 \alpha_{XZ}^{BC} \geq -\log_2 \alpha_{XZ}^{BC}$$

Complementary observables and $\boldsymbol{d_B = d_C = d}$

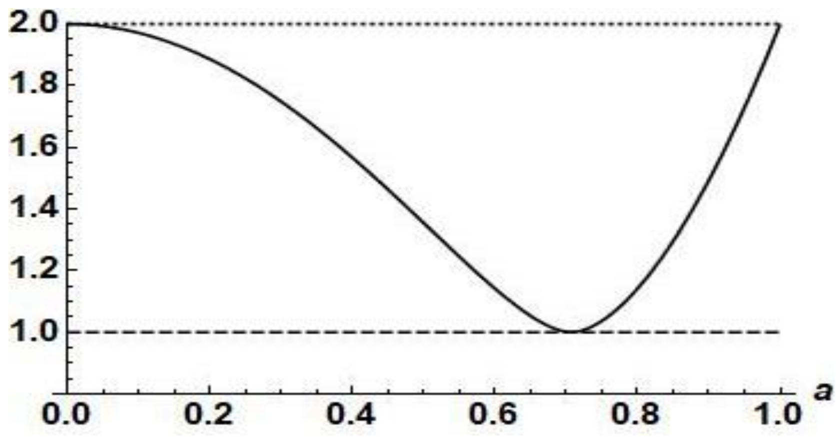
$$H(X_B, X_C|X_A) + H(Z_B, Z_C|Z_A) \geq 2\log_2 d$$

Any state that violates this inequality must be steerable from Alice to Bob and Charlie!

Steering detection (d=2)

$$H(X_B, X_C|X_A) + H(Z_B, Z_C|Z_A) \geq 2$$

(A) GHZ states: $|\psi\rangle = a|000\rangle + \sqrt{1-a^2}\,|111\rangle$



Steerable for all $a \in (0,1)$

(B) Resistance to white noise: $\rho = p\rho_{GHZ} + \frac{1-p}{8} I$ detect as steerable for $p > 0.80$

(C) W state: $|\psi\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ detects as steerable

Genuine steering inequality

Define: $A(O_A, O_B, O_c) = H(O_B, O_C | O_A) + H(O_B | O_A, O_C) + H(O_C | O_A, O_B)$

Non-genuine tripartite steerable state →

$$\sum_{O=X,Z} A(O_A, O_B, O_c) \geq -2\log_2 \alpha_{XZ}^{BC} = \log_2 d_{BC}$$

Complementary Observables

Any state that violates this inequality is identified as genuine tripartite steerable from Alice to Bob and Charlie!

Nonsteerable state →

$$\sum_{O=X,Z} A(O_A, O_B, O_c) \geq -\log_2 \alpha_{XZ}^{BC} = 2\log_2 d_{BC}$$

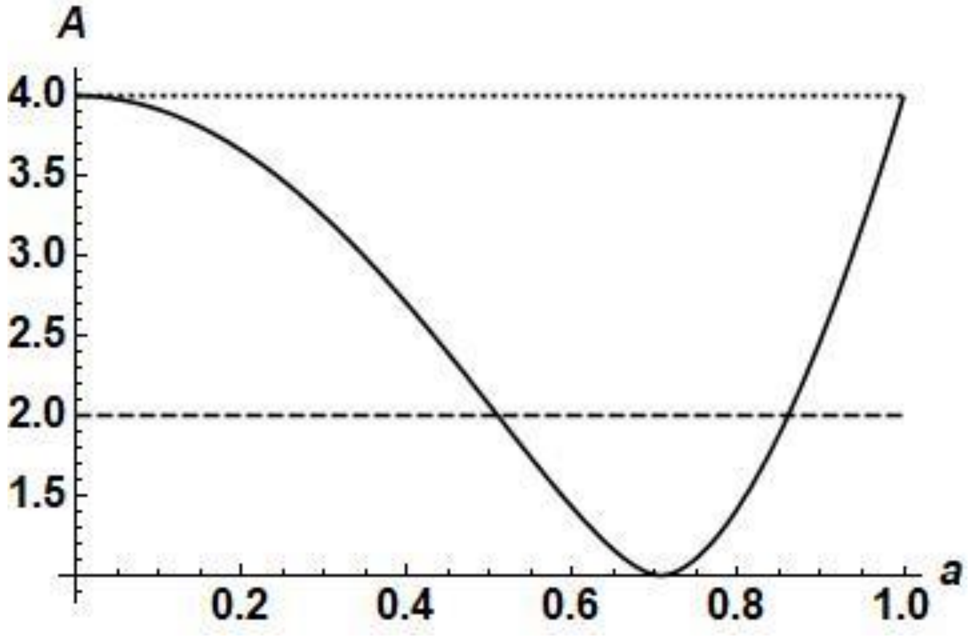
Genuine steering detection (d=2)

A=$\sum_{X,Z} A(O_A, O_B, O_c) \geq 4$

(A) GHZ states: $|\psi\rangle = a|000\rangle + \sqrt{1-a^2}\,|111\rangle$

$A < 4$: **tripartite steerable states**

$A < 2$: **genuine steerable states**



(B) Resistance to white noise: $\rho = p\rho_{GHZ} + \frac{1-p}{8} I$

Genuine steerable for $p > 0.95$

(SDP method: $p > 0.54$)

(C) W state: $|\psi\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

It cannot be identified as genuine tripartite steerable with this criterion.

Perspectives

- Find stronger steering inequalities based on entropic uncertainty with quantum memories. Investigate the relation with monogamy properties of steering.

- Analyze multipartite steering with generalized entropies (Tsallis, Renyi).

- Multipartite steering criteria based on universal uncertainty relations (Schur-concave functions as uncertainty quantifiers).

- Study nonlocal correlations within the framework of entropic uncertainty relations.

Summary

(i) We have seen that there exist at least **three types of quantum correlations.** Each type can enable to perform some particular processing task.

(ii) **The framework of EUR allows to study different types of quantum correlations**.

(iii) **We introduced state-independent inequalities whose violations certifies the different levels of multipartite steering**, namely «simple» and genuine multipartite steering. They can be used for steering detection given an unknown quantum state.

A. Riccardi, C. Macchiavello and L. Maccone, Multipartite steering inequalities based on entropic uncertainty relations, **Phys. Rev. A 97, 052307 (2018)**

State-dependent inequality

Entropic uncertainty relations with quantum memory:

$$H(X_B|X_A) + H(Z_B|Z_A) \geq -\log_2 \alpha_{XZ}^{B} + S(B|A)$$

$$H(X_C|X_A) + H(Z_C|Z_A) \geq -\log_2 \alpha_{XZ}^{C} + S(C|A)$$

Quantum conditional Von Neumann entropies

Non-genuine tripartite steerable state

⇓

$$\sum_{XZ} H(O_B, O_C|O_A) \geq -\log_2 \alpha_{XZ}^{BC} + \sum_{\gamma} q_B(\gamma) S_\gamma(B|A) + \sum_{\omega} q_C(\omega) S_\omega(C|A)$$

Bipartite steering

We say that **the correlations demonstrate quantum steering** iff:

(1) For every possible measurements X_A and X_B the joint measurement probabilities cannot be written as:

$$p(x_a, x_b) \neq \sum_{\lambda} q(\lambda) p(x_a|\lambda) p_Q(x_b|\lambda)$$

Local Hidden State Model

$p_Q(x_b|\lambda)$ is originating from pre-determined state ρ_λ^B, that depends on the classical variable λ. Bob can explain its results in terms of local states.

(2) Entanglement between Alice and Bob is detected.

H.M. Wiseman et al. Phys. Rev. Lett. 98, 140402 (2007)

Two-to-one steering scenario

A → C ← B

Tripartite nonsteerable state from Alice and Bob to Charlie. →

$$\sum_{X,Z} H(O_C\,|O_A) + H(O_C\,|O_B) \geq -2\log_2 \alpha_{XZ}^{C}$$

Genuine tripartite nonsteerable state from Alice and Bob to Charlie. →

$$\sum_{X,Z} H(O_C\,|O_A) + H(O_C\,|O_B) \geq -\log_2 \alpha_{XZ}^{C}$$

Sufficient conditions to tripartite steering and genuine tripartite steering.

Genuine steering inequality (1)

(1) For any non genuine tripartite steerable states we have:

$$H(X_B, X_C|X_A) \geq \sum_{\nu} q_A(\nu) H_\nu(X_B, X_C) + \sum_{\gamma} q_B(\gamma) \left[H_\gamma(X_B) + H_\nu(X_C|X_A)\right]$$

$$+ \sum_{\omega} q_C(\omega) \left[H_\omega(X_C) + H_\omega(X_B|X_A)\right]$$

Genuine steering inequality (1)

(1) For any non genuine tripartite steerable states we have:

$$H(X_B, X_C|X_A) \geq \sum_{\nu} q_A(\nu) H_\nu(X_B, X_C) + \sum_{\gamma} q_B(\gamma) \left[H_\gamma(X_B) + H_\gamma(X_C|X_A)\right] + \sum_{\omega} q_C(\omega) \left[H_\omega(X_C) + H_\omega(X_B|X_A)\right]$$

$H_\gamma(X_C|X_A) \geq 0$

$H_\omega(X_B|X_A) \geq 0$

Genuine steering inequality (1)

(2) For any non genuine tripartite steerable states we have:

$$H(X_B, X_C|X_A) \geq \sum_\nu q_A(\nu) H_\nu(X_B, X_C) + \sum_\gamma q_B(\gamma)\, H_\gamma(X_B) + \sum_\omega q_C(\omega) H_\omega(X_C)$$

If we use directly the entropic uncertainty relations, we would not find a state-indepedent bound.

Example: $\boldsymbol{X}, \boldsymbol{Z}$ complementary + $\boldsymbol{d_B = d_C = d}$

$$H(X_B, X_C|X_A) + H(Z_B, Z_C|Z_A) \geq 2\log_2 d \sum_\nu q_A(\nu) + \log_2 d \sum_\gamma q_B(\gamma) + \log_2 d \sum_\omega q_C(\omega)$$

Genuine steering inequality (2)

(3) Any non-genuine tripartite steerable state satisifes also:

$$H(X_B|X_A, X_C) \geq \sum_{\gamma} q_B(\gamma)\, H_{\gamma}(X_B)$$

$$H(X_B|X_A, X_C) \geq \sum_{\omega} q_C(\omega)\, H_{\omega}(X_C)$$

(4) Define: $A(O_A, O_B, O_c) = H(O_B, O_C|O_A) + H(O_B\,|O_A\,, O_C) + H(O_C\,|O_A\,, O_B)$

$$A(X_A, X_B, X_C) + A(Z_A, Z_B, Z_C) \geq 2\log_2 d \sum_{\nu} q_A(\nu) + 2\log_2 d \sum_{\gamma} q_B(\gamma) + 2\log_2 d \sum_{\omega} q_C(\omega)$$

$$\geq 2\log_2 d$$

EUR approach to quantum correlations

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