Generalized measurements, quantum designs and ‘quantum simplices’

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A generalized (discrete) quantum measurement:
POVM (positive operator valued measure) $\Pi = \{\Pi_j\}_{j=1}^k$, where

$$\Pi_j \in \mathcal{L}(\mathbb{C}^d), \quad \Pi_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{k} \Pi_j = \mathbb{I}.$$

For pre-measurement state ρ, the probability of j-th outcome is $\mathrm{tr}(\rho\Pi_j)$.

How to quantify the indeterminacy of the measurement?

The **Shannon entropy of POVM** Π is defined by:

$$H_\Pi(\rho) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho\Pi_j)),$$

for an initial state ρ, where $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

What are we looking for:

$$\min_{\tau \in \mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau) \quad \text{and} \quad \max_{\tau \in \mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau).$$

But that is the same as:

$$\min_{P \in \Delta^1_\Pi} H(P) \quad \text{and} \quad \max_{P \in \Delta^1_\Pi} H(P),$$

where $\Delta^1_\Pi := \{P \in \Delta_k : P = (\mathrm{tr}(\rho\Pi_1), \ldots, \mathrm{tr}(\rho\Pi_k)) \text{ for some } \rho \in \mathcal{P}(\mathbb{C}^d)\}$.

Let

$$\Delta_\Pi := \{P \in \Delta_k : P = (\mathrm{tr}(\rho\Pi_1), \ldots, \mathrm{tr}(\rho\Pi_k)) \text{ for some } \rho \in \mathcal{S}(\mathbb{C}^d)\} = \mathrm{conv}(\Delta^1_\Pi).$$

We call it the set of **'allowed' probabilities**.

How do Δ_Π and Δ^1_Π look like?

How do Δ_Π and Δ_Π^1 look like?

- $\Delta_\Pi \subset \Delta_k$, $\dim \Delta_\Pi \leq d^2 - 1$ with equality iff Π is informationally complete
- Simplest case: Π is a rank-one PVM (projective valued measure). Then

$$\Delta_\Pi = \Delta_\Pi^1 = \Delta_k = \Delta_d .$$

- In general: **joint algebraic numerical range** of $(\Pi_1, \ldots, \Pi_k)$
- In dimension 2: Δ_Π^1 is one of the following: a sphere, an ellipsoid, a disc, an ellipse (with the interior), a segment or a point.

When Δ_Π is 'the same' as $\mathcal{S}(\mathbb{C}^d)$?

For $A, B \in \mathcal{L}_s(\mathbb{C}^d)$ we denote their Hilbert-Schmidt inner product by $\langle\langle A|B\rangle\rangle$.

We say that Δ_Π is isomorphic to quantum state space $\mathcal{S}(\mathbb{C}^d)$ if there exists $\alpha > 0$ such that

$$\langle\langle \rho - I/d|\sigma - I/d\rangle\rangle = \alpha\langle p(\rho) - c, p(\sigma) - c\rangle \quad \forall \rho, \sigma \in \mathcal{S}(\mathbb{C}^d), \tag{1}$$

where $p(\rho) = (\mathrm{tr}(\rho\Pi_1), \ldots, \mathrm{tr}(\rho\Pi_k))$ and $c = p(I/d) = (\mathrm{tr}\Pi_1/d, \ldots, \mathrm{tr}\Pi_k/d)$.

Theorem

Δ_Π is isomorphic to quantum state space $\mathcal{S}(\mathbb{C}^d)$ iff $\{\Pi_i - (\mathrm{tr}\Pi_i/d)I\}_{i=1}^k$ is a **tight frame** *in $\mathcal{L}_s^0(\mathbb{C}^d)$. Then $\alpha = (d^2 - 1)/(\sum_{i=1}^k(\mathrm{tr}\Pi_i^2 - (\mathrm{tr}\Pi_i)^2/d))$.*

Frames, tight frames

V – N-dimensional Hilbert space with inner product $\langle\cdot|-\rangle$
$F := \{f_1, \ldots, f_m\} \subset V$

F is a **frame** if there exist $\alpha, \beta > 0$ such that

$$\alpha\|v\|^2 \leq \sum_{i=1}^{m} |\langle v|f_i\rangle|^2 \leq \beta\|v\|^2 \quad \text{for all } v \in V.$$

F is called a **tight frame** if $\alpha = \beta$.

An operator $S := \sum_{i=1}^{m} |f_i\rangle\langle f_i|$ is called a **frame operator**.

F is a tight frame iff $S = \alpha I$.

$\mathcal{L}_s(\mathbb{C}^d)$ – d^2-dimensional real Hilbert space of selfadjoint operators on $\mathbb{C}^d$ with the Hilbert-Schmidt inner product $\langle\langle A|B\rangle\rangle = \mathrm{tr}(AB)$

$\mathcal{L}_s^0(\mathbb{C}^d) := \{A \in \mathcal{L}_s(\mathbb{C}^d) | \mathrm{tr}A = 0\}$ – $(d^2 - 1)$-dimensional subspace of traceless operators

$$\pi_0 : \mathcal{L}_s(\mathbb{C}^d) \ni A \mapsto A - (\mathrm{tr}A/d)I \in \mathcal{L}_s(\mathbb{C}^d)$$

π_0 is the orthogonal projection onto $\mathcal{L}_s^0(\mathbb{C}^d)$

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Theorem (reformulated)

The following conditions are equivalent:

- *Δ_Π is isomorphic to quantum state space $\mathcal{S}(\mathbb{C}^d)$*
- *$\pi_0(\Pi)$ is a* **tight frame** *in $\mathcal{L}_s^0(\mathbb{C}^d)$ with frame constant $1/\alpha$*
- $\sum_{i=1}^k |\pi_o(\Pi_i)\rangle\rangle\langle\langle\pi_0(\Pi_i)| = \frac{1}{\alpha}\mathcal{I}_0$.

Moreover, $\alpha = (d^2 - 1)/(\sum_{i=1}^k (\mathrm{tr}\Pi_i^2 - (\mathrm{tr}\Pi_i)^2/d))$.

Some special cases of POVMs Π such that $\pi_0(\Pi)$ is a tight frame in $\mathcal{L}_s^0(\mathbb{C}^d)$:

- $\mathrm{tr}\Pi_i = d/k$ for $i = 1, \ldots, k$. Then

$$\sum_{i=1}^{k} |\Pi_i\rangle\rangle\langle\langle\Pi_i| = \frac{1}{\alpha}\mathcal{I} + \frac{\alpha d - k}{\alpha dk}|I\rangle\rangle\langle\langle I|$$

  In particular, Π is a **tight IC-POVM** [Scott '06]

- $\mathrm{tr}\Pi_i = d/k$ **and** $\mathrm{rank}\Pi_i = 1$ for $i = 1, \ldots, k$. Then Π is a **2-design POVM**, i.e. pure states $\rho_i := \frac{k}{d}\Pi_i$ ($i = 1, \ldots, k$) are such that

$$\frac{1}{k^2}\sum_{j,l=1}^{k} f(\mathrm{tr}(\rho_j\rho_l)) = \iint_{\mathcal{P}(\mathbb{C}^d)\times\mathcal{P}(\mathbb{C}^d)} f(\mathrm{tr}(\rho\sigma))d\mu(\rho)d\mu(\sigma)$$

  for every $f : \mathbb{R} \to \mathbb{R}$ polynomial of degree t or less, where μ denotes the unique unitarily invariant measure on $\mathcal{P}(\mathbb{C}^d)$.
  Examples: SIC-POVMs, complete MUBs

- $\{\Pi_j\}_{j=1}^{k}$ is a 2-design POVM iff

$$\tau = (d+1)\sum_{j=1}^{k}\frac{k}{d}\mathrm{tr}(\tau\Pi_j)\Pi_j - I$$

for every τ such that $\tau^* = \tau$ and $\mathrm{tr}\tau = 1$.

- For 2-design POVM Π, $\rho_j = \frac{k}{d}\Pi_j$ and $p_j(\tau) = \mathrm{tr}(\tau\Pi_j)$ we thus have

$$\sum_{j=1}^{k} p_j(\tau) = 1 \qquad (2)$$

$$\sum_{j=1}^{k}(p_j(\tau))^2 = \frac{d(\mathrm{tr}(\tau^2)+1)}{k(d+1)} \qquad (3)$$

$$\sum_{j,l,m=1}^{k} p_j(\tau)p_l(\tau)p_m(\tau)\mathrm{tr}(\rho_j\rho_l\rho_m) = \frac{\mathrm{tr}(\tau+I)^3}{(d+1)^3} \qquad (4)$$

Remark: A self-adjoint operator τ is a pure quantum state iff $\mathrm{tr}\tau = 1$, $\mathrm{tr}\tau^2 = 1$ and $\mathrm{tr}\tau^3 = 1$.

Theorem

Let Π be a 2-design POVM and let $\rho_j = \frac{k}{d}\Pi_j$ ($j = 1, \ldots, k$). Let $(p_1, \ldots, p_k) \in \mathbb{R}^k$ be such that

- $\frac{k}{d}p_l = (d+1)\sum_{j=1}^k p_j \mathrm{tr}(\rho_j \rho_l) - 1$ *for* $l = 1, \ldots, k$,
- $\sum_{j=1}^k p_j^2 = \frac{2d}{k(d+1)}$,
- $\sum_{j,l,m=1}^k p_j p_l p_m \mathrm{tr}(\rho_j \rho_l \rho_m) = \frac{d+7}{(d+1)^3}$.

Then $(p_1, \ldots, p_k) \in \Delta_k$ and $p_l = \frac{d}{k}\mathrm{tr}(\rho_l \tau)$, where $\tau := (d+1)\sum_{j=1}^k p_j \rho_j - I$, $\tau \in \mathcal{P}(\mathbb{C}^d)$.

$$\Delta_k^{(2)} := \{P \in \Delta_k | \sum_{j=1}^{k} p_j^2 = \frac{2d}{k(d+1)}\}.$$

This value is referred to as the **index of coincidence** and denoted $IC(P)$. Then [Harremoes&Topsoe '01]:

$$\arg\max_{P\in\Delta_k^{(2)}} H(P) = \left\{ s\left(p, \frac{1-p}{k-1}, \ldots, \frac{1-p}{k-1}\right) | s \in S_k \right\},$$

where $p = \frac{1}{k}\left(1 + \sqrt{\frac{(d-1)(k-1)}{d+1}}\right)$.

and:

$$\arg\min_{P\in\Delta_k^{(2)}} H(P) = \{s(p, \underbrace{q, \ldots, q}_{m}, \underbrace{0, \ldots, 0}_{k-m-1}) | s \in S_k\},$$

where $p = \frac{1-\sqrt{r_m}}{m+1}$, $q = \frac{m+\sqrt{r_m}}{m(m+1)}$, $m = \lfloor (IC(P))^{-1} \rfloor$ and $r_m = \frac{IC(P) - \frac{1}{m+1}}{\frac{1}{m} - \frac{1}{m+1}}$.

For a 2-design POVM Π we have $\Delta_\Pi \subset \Delta_k^{(2)}$ and thus we get the following bounds on H_Π:

$$\min_{P\in\Delta_k^{(2)}} H(P) \leq \min_{P\in\Delta_\Pi} H(P) = \min_{\tau\in\mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau)$$

$$\max_{P\in\Delta_k^{(2)}} H(P) \geq \max_{P\in\Delta_\Pi} H(P) = \max_{\tau\in\mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau)$$

For which 2-design POVMs these bounds are met?

In other words:
for which 2 design POVM Π there exist $\tau_{\min}, \tau_{\max} \in \mathcal{P}(\mathbb{C}^d)$ such that

$$P(\tau_{\min}) \in \arg \min_{P\in\Delta_k^{(2)}} H(P)$$

and

$$P(\tau_{\max}) \in \arg \max_{P\in\Delta_k^{(2)}} H(P).$$

Upper bound

Theorem

Let Π *be a 2-design POVM in* $\mathbb{C}^d$*. Then*

$$\max_{P\in\Delta_k^{(2)}} H(P) = \max_{P\in\Delta_\Pi} H(P) = \max_{\tau\in\mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau)$$

if and only if Π *is a* **SIC-POVM***.*

Sketch of the proof.

- For a SIC-POVM $\Pi = \{\frac{1}{d}\rho_j\}_{j=1}^{d^2}$ take $\tau = \rho_j$.
- To see the converse it is enough to show that the equality holds iff $k = d^2$.
- Use the fact that $\sum_{j=2}^k \rho_j = \frac{k}{d}I - \rho_1$ and the condition $\frac{k}{d}p_l = (d+1)\sum_{j=1}^k p_j \mathrm{tr}(\rho_j\rho_l) - 1$ for $l = 1, \ldots, k$. □

Lower bound

Known examples for which the lower bound **is** achieved:

- the unique (tetrahedral) **SIC-POVM** in dimension 2
  - minimal configuration: 'twin' SIC
- the **supersymmetric SIC-POVM** in dimension 3 (the Hesse configuration)
  - minimal configuration: complete set of MUB
- a **generic SIC-POVM** in dimension 3
  - minimal configuration: an orthonormal basis
- the **complete set of MUB** in dimension 3
  - minimal configuration: the supersymmetric SIC
- the **supersymmetric SIC-POVM** in dimension 8 (the Hoggar lines)
  - minimal configuration: 'twin' SIC

Known examples for which the lower bound is **not** achieved:

- SIC-POVM in dimension 4
- complete sets of MUBs in even dimensions
- configuration of 45 vectors in $\mathbb{C}^5$ (Example 18 in [Hoggar '82])

Note that for all known examples with attained lower bound we have:

$$IC(P)^{-1} = \frac{k(d+1)}{2d} \in \mathbb{N}$$

If $m := (IC(P))^{-1} = \frac{k(d+1)}{2d} \in \mathbb{N}$, then

$$\arg \min_{P \in \Delta_k^{(2)}} H(P) = \{s(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m}, \underbrace{0, \ldots, 0}_{k-m}) | s \in S_k\}.$$

Let $\Pi = \{\frac{d}{k}\rho_j\}_{j=1}^{k}$ be a 2-design POVM in $\mathbb{C}^d$ and let $\frac{k(d+1)}{2d} \in \mathbb{N}$. Then the following conditions are equivalent:

- $\min_{P \in \Delta_k^{(2)}} H(P) = \min_{P \in \Delta_\Pi} H(P) = \min_{\tau \in \mathcal{P}(\mathbb{C}^d)} H_\Pi(\tau)$
- There exists $J \subset \{1, 2, \ldots, k\}$ such that $|J| = \frac{k(d-1)}{2d}$ and $\frac{2d}{k} \sum_{j \in J} \rho_j$ is an orthogonal projection onto $(d-1)$-dimensional supspace of $\mathbb{C}^d$.
- Let $\rho_j = |\psi_j\rangle\langle\psi_j|$ for some $\psi_j \in \mathbb{C}^d$. Then there exists $J \subset \{1, 2, \ldots, k\}$ such that $|J| = \frac{k(d-1)}{2d}$, $\dim(\mathrm{span}\{\psi_j\}_{j \in J}) = d - 1$ and $\{\psi_j\}_{j \in J}$ is a normalized tight frame in $\mathrm{span}\{\psi_j\}_{j \in J}$.
- There exists $J \subset \{1, 2, \ldots, k\}$ such that $|J| = \frac{k(d-1)}{2d}$ and $\sum_{j \in J} p_j(\tau) \leq \frac{1}{2}$ for all $\tau \in \mathcal{P}(\mathbb{C}^d)$.

Quantum bayesianizm [Appleby, Fuchs, Stacey & Zhu, 2016]

- "Bureau of Standards" measurement: SIC-POVM $E = \{E_j\}_{j=1}^{d^2}$
- an arbitrary POVM $F = \{F_j\}_{j=1}^{m}$
- let $r(j|i) = \mathrm{tr}(E_i F_j)/d$
- for $\tau \in \mathcal{S}(\mathbb{C}^d)$ let $p_\tau(j) = \mathrm{tr}(\tau E_j)$ and $q_\tau(j) = \mathrm{tr}(\tau F_j)$.
- **URGLEICHUNG**:

$$q_\tau(j) = \sum_{i=1}^{d^2} (\alpha p_\tau(i) - \beta) r(j|i)$$

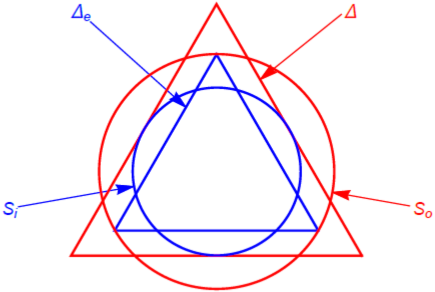
  with $\alpha = d + 1$ and $\beta = 1/d$.
- **QPLEX**$= \{(p_\tau(1), \dots, p_\tau(d^2)) | \tau \in \mathcal{S}(\mathbb{C}^d)\}$
- BUT for any 2-design POVM $E = \{E_j\}_{j=1}^{k}$ and $r(j|i) = \mathrm{tr}(E_i F_j)(d/k)$ we also have

$$q_\tau(j) = \sum_{i=1}^{k} (\alpha p_\tau(i) - \beta) r(j|i)$$

  with $\alpha = d + 1$ and $\beta = d/k$

Properties of qplex:

- contained in outer ball, contains inner ball
- the same as quantum state space
- Can geometrical properties and abstract definitions be transferred into general 2-design POVM case?



- How general qplex can be located in the large simplex?
- Can this better understanding of the structure of Δ_Π lead to progress in the minimal entropy problem?

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