Designs for mixed quantum states

J. Czartowski[1] D. Goyeneche[1,3] M. Grassl[4]
K. Życzkowski[1,2]

[1]Faculty of Physics, Astronomy and Computer Sciences,
Jagiellonian University, Cracow, Poland
[2]Center for Theoretical Physics,
Polish Academy of Sciences, Warsaw, Poland
[3]Departamento de Física, Facultad de Ciencias Básicas,
Universidad de Antofagasta, Antofagasta, Chile
[4]Max-Planck-Institut für die Physik des Lichts
Erlangen, Germany

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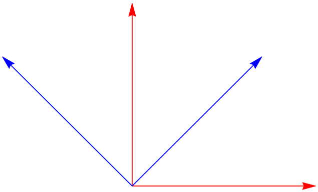
Outline

Mutually Unbiased Bases

- Mutually unbiased bases (MUB) are such orthogonal bases on $\mathcal{H}^n$ $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ that

$$|\langle\psi_i|\phi_j\rangle|^2 = \frac{1}{n} \qquad (1)$$

- Such bases provide maximally different quantum measurements.
- For Hilbert space of dimension n we can have at most $n+1$ such bases.

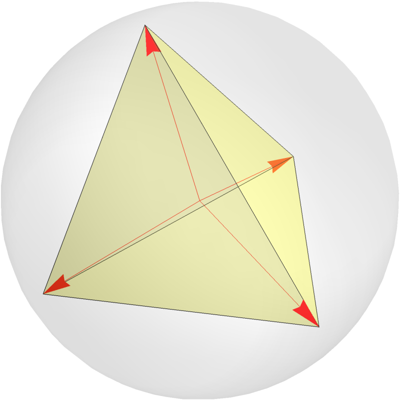


Symmetric Informationally Complete POVM

- Symmetric informationally complete (SIC) POVM is such a set of n^2 vectors $\{|\psi_i\rangle\}$ on $\mathcal{H}^n$, that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n+1} \qquad (2)$$

- They may be thought as equiangular structures on Hilbert space.



Complex projective t-designs

Definition

Any ensemble $\{|\psi_i\rangle\}_{i=1}^{M}$ of pure states in $\mathcal{H}^n$ is called **complex projective t-design** if for any polynomial f_t of degree at most t in both components of the states and their conjugates the average over the ensemble is the same as average over the whole space $\mathcal{H}^n$:

$$\frac{1}{M}\sum_{i=1}^{M} f_t(\psi_i) = \int_{\mathcal{H}^n} f_T(\psi)\,\mathrm{d}\psi_{Haar}\,. \tag{3}$$

- Complex projective t-designs are used for quantum state tomography, quantum fingerprinting and quantum cryptography.
- Examples include mutually unbiased bases (MUB) and symmetric informationally complete (SIC) POVM.

Interesting case - isoentangled SIC-POVM

- Equation (3) sets a condition for average entanglement of vectors in a 2-design in $\mathcal{H}^n \otimes \mathcal{H}^n$

$$\left\langle \mathrm{Tr}\left[\left(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i| \right)^2 \right] \right\rangle = \frac{2n}{n^2+1} \tag{4}$$

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- Zhu et al. found a particularly interesting example of SIC for $n = 2$, such that entanglement is constant.

$$\forall_i \ \mathrm{Tr}\left[\left(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i| \right)^2 \right] = \frac{4}{5} \tag{5}$$

Isoentangled MUB for 2 qubits?

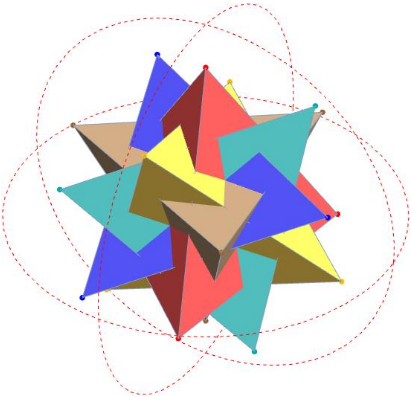
Question:

Isoentangled MUB for 2 qubits?

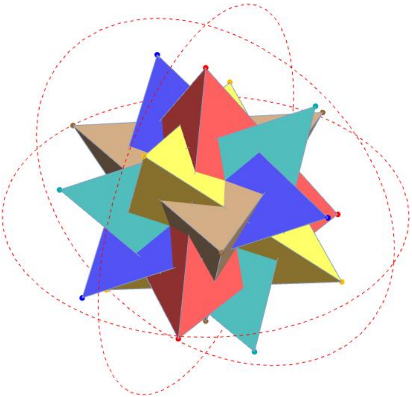
Question:

Can we have similar configuration of MUB for 2 qubits?

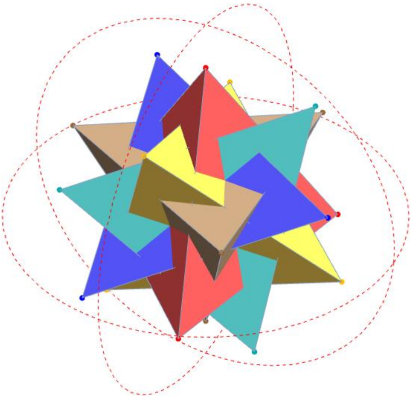
The answer is positive!



- Each basis is represented by a tetrahedron in a Bloch sphere.



- Each basis is represented by a tetrahedron in a Bloch sphere.
- The whole set's convex hull is dodecahedron. All the tetrahedra form a **regular 5-tetrahedra compound**.



- Each basis is represented by a tetrahedron in a Bloch sphere.
- The whole set's convex hull is dodecahedron. All the tetrahedra form a **regular 5-tetrahedra compound**.
- Analytic form is not simple.

Mixed states t-designs

Definition

Any ensemble $\{\rho_i\}_{i=1}^{M}$ of M density matrices is called a **mixed states t-design** if for any polynomial g_t of degree t in the eigenvalues λ_j of the state ρ the average over the ensemble is equal to the mean value over the space of mixed states Ω_N with respect to the flat Hilbert-Schmidt measure:

$$\frac{1}{M}\sum_{i=1}^{M} g_t(\rho_i) = \int_{\Omega_N} g_t(\rho)\,\mathrm{d}\rho_{HS}\,. \tag{6}$$

Method of obtaining

Proposition

Any complex projective t-design $\{|\psi_i\rangle\}_{i=1}^{M}$ in the composite Hilbert's space $\mathcal{H}_{NA} \otimes \mathcal{H}_{NB}$ induces, by partial trace, a mixed quantum states design $\{\rho_i\}_{i=1}^{M}$ in Ω_N with $\rho_i = \mathrm{Tr}_B |\psi_i\rangle\langle\psi_i|$. The same property holds also for the dual set $\{\rho_i' : \rho_i' = \mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|\}$.

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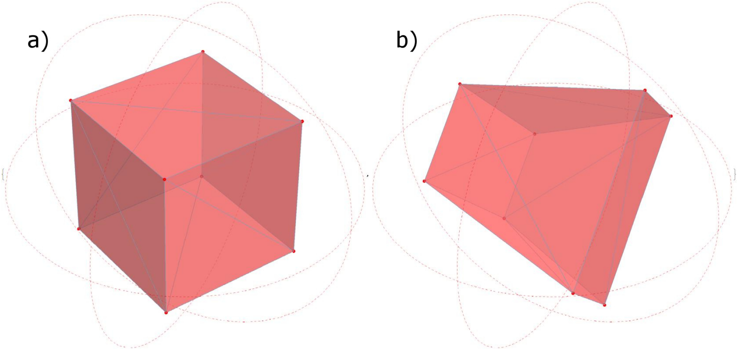
Observation

Every positive operator-valued measurement (POVM) induces a mixed states 1-design.

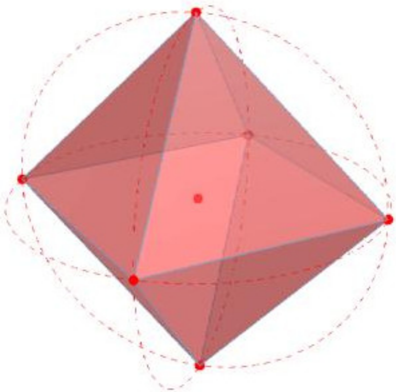
Isoentangled SIC-POVM for 2 qbits[1]



- In Alice reduction SIC-POVM yields Platonic solid - a cube. In Bob reduction constellation isn't as regular as for Alice.

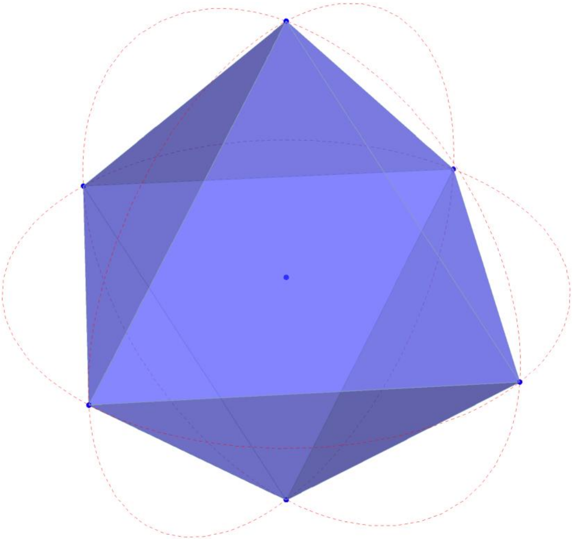
[1] H. Zhu, B. Englert, Phys. Rev. A 84, 022327 (2011)

2-qbit MUB

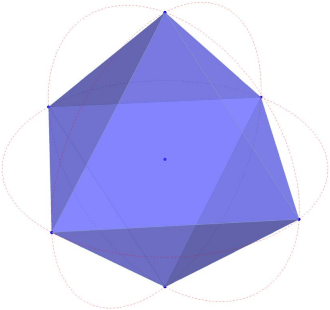


- Reduced states ρ_A and ρ_B form vertices of a regular octahedron within the Bloch ball with additional maximally mixed state in the centre.

Uncanny similarity



Uncanny similarity - Hoggar example 24[2]



- Hoggar provides an example of projective 3-design in $\mathcal{H}^4$ attained by considering particular complex polytope.

[2] Hoggar S 1998 Geometriae Dedicata 69, 287–289

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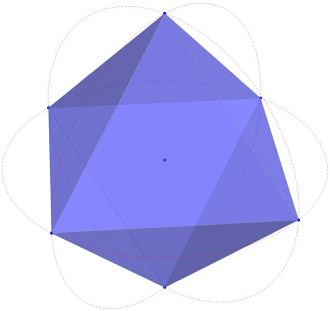

- Hoggar provides an example of projective 3-design in $\mathcal{H}^4$ attained by considering particular complex polytope.
- Within both reductions we observe exact identity with MUB for 2 qubits.

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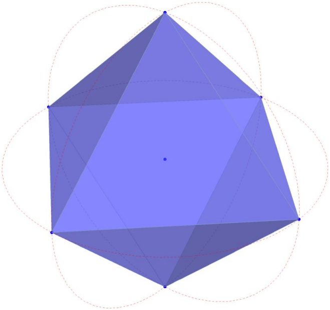


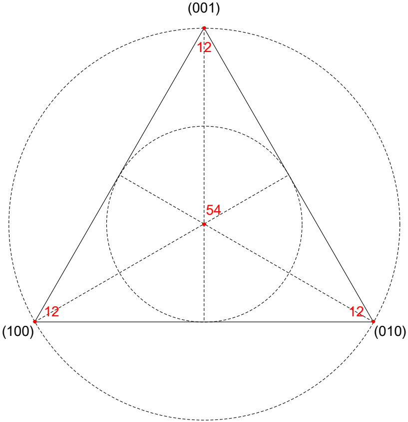
- Hoggar provides an example of projective 3-design in $\mathcal{H}^4$ attained by considering particular complex polytope.
- Within both reductions we observe exact identity with MUB for 2 qubits.
- Thus we conclude that **MUB for 2 qubits induces mixed 3-design**.

[2]Hoggar S 1998 Geometriae Dedicata 69, 287–289

Qutrits

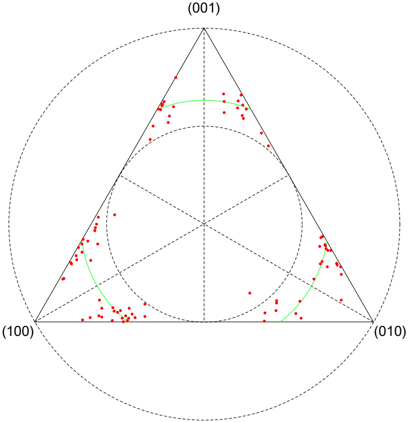
But what can be said for qutrits?

Standard representation of 2 qutrits MUB



- Standard representation of MUB for 2 qtrits consists in 4 separable and 5 maximally entangled bases.
- Representation of density matrices in the triangle of eigenvalues proves to be the most comprehensible.

Numerical results for 2 qutrits isoentangled MUB



- Numerical search for isoentangled MUB for 2 q-trits were inconclusive.
- However, all the points seem to converge at the green circle - subset of density matrices with analytically calculated entanglement.

Summary

- We introduce, by analogy to complex projective t-designs, a concept of mixed states t-designs and provide a way to obtain such constellations of mixed states.
- We show examples based on already known projective t-designs and analyze them for **regularities in Bloch ball**
- Most importantly, we introduce a novel set of isoentangled MUBs for 2 qubits.

- Open questions
  - Analytic isoentangled structures for qutrit-qutrit?
  - Sufficient conditions for mixed states t-designs?

Thank you
for your attention!

For Further Reading

- A. J. Scott, Tight informationally complete quantum measurements, J. Phys. A 39, 13507 (2006)
- S. Hoggar, Geometriae Dedicata 69, 287–289 (1998)
- H. Zhu, B. Englert, Phys. Rev. A 84, 022327 (2011)

Design-like structure

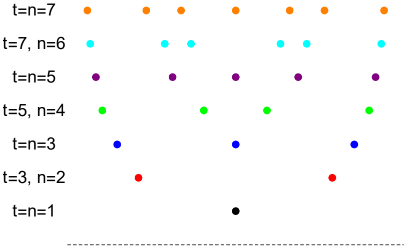
- Consider a measure $\mu(x)$ defined on the interval $[0, 1]$ and a minimal sequence of points $\{x_i : x_i \in [0, 1]\}_{i=1}^{M}$ such that

$$\frac{1}{M}\sum_{i=1}^{M} x_i^t = \int_0^1 x^t \mu(x)\,\mathrm{d}x\,. \tag{7}$$

- Such structures may find use in approximate integration using Tailor expansion

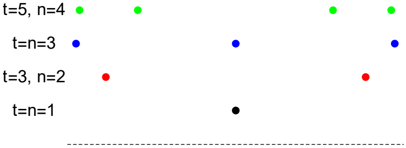
$$\int_0^1 f(x)\,\mathrm{d}x = \left(\sum_{i=0}^{t}\sum_{j=1}^{M}\frac{1}{i!}\frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i}\Big|_{x=x_0}(x_j - x_0)^t\right) + O(x^{t+1}) \tag{8}$$

Flat measure on a line



- $\mu(x) = 1$ defines flat measure.
- Configurations have been found up to $t = 7$.
- Limitation - *computation power!*

HS measure - qubit example

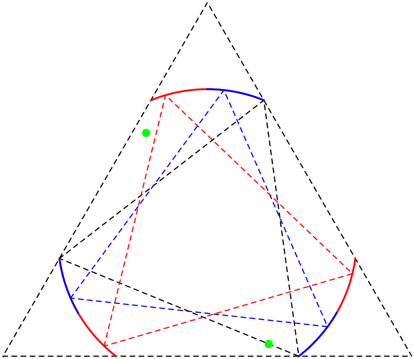


- Let us consider Hilbert-Schmidt measure on eigenvalues of density matrices

$$\mu_{HS}(x) = 3(2x-1)^2 \quad (9)$$

- For $t = 5$ we found $n = 4$ points.
- Any higher n or t results in complex solutions

Extension for qutrits?



- For qutrits measure $\mu(x, y)$ is described on a triangle of eigenvalues.
- For $t = 1$ a pair of points (ex. in green) can be easily found.
- For $t = 2$ there exists a continuous family of triplets that satisfy the given conditions.
- This suggests significant difficulties in obtaining isoentangled MUBfor $\mathcal{H}^9$