Hybrid quantum computing algorithm for solving hard graph problems

Wojtek Burkot

QM Basics

1) Suzuki-Totter expansion
2) Unitary time evolution of a state | ϕ> in Hilbert space (perfect linearity):

$U|\phi(t)\rangle = |\phi(t'=t+dt)\rangle = \sum_i a_i |\phi_i\rangle$, where $\sum_i |a_i|^2=1$

3) Von Neumman (projection) measurement - state after the measurement <ϕ|H|ϕ> of an Hermitean observable is in a subspace spanned on $|\phi_{i \in in}\rangle$, consistent with the measurement result , with the probability $\sum_{i \in in} |a_{i \in in}|^2$

Consequences for QC

qbit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$

quantum register: $|\Psi\rangle = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes \ldots \otimes |\psi\rangle_n$

:-) : exponential capacity N= 2^n , extreme SIMD

:-(: exponentially low probability of ending in the desired state by brute force (but quadratically better than classically)

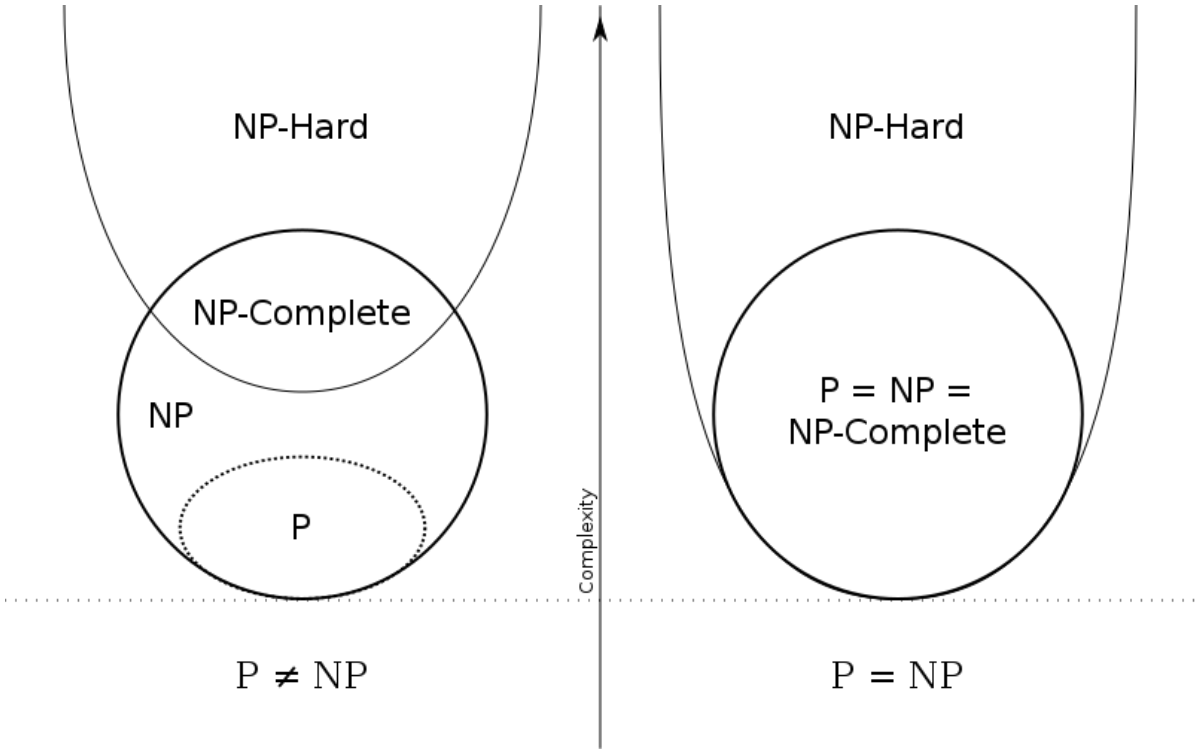
:-(: no cloning, prohibiting backtracking

:-(: fragility, leading to losing the state to decoherence

- **State of the art**

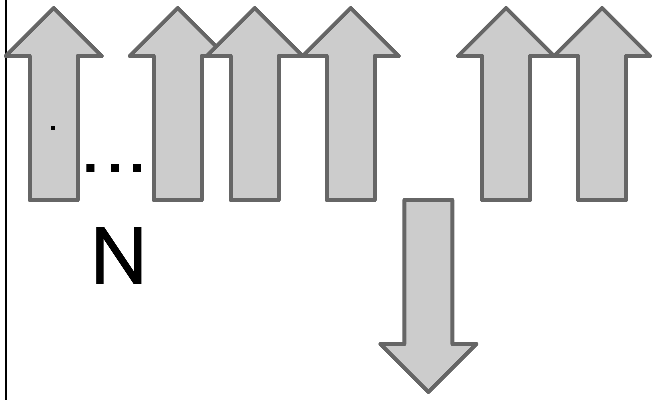
- Algorithms: Simon, Grover, Shor
- Decoherence fighting: quantum error correction, Zeno effect, mixed state computing,
- Measurement based computing
- Quantum Annealing

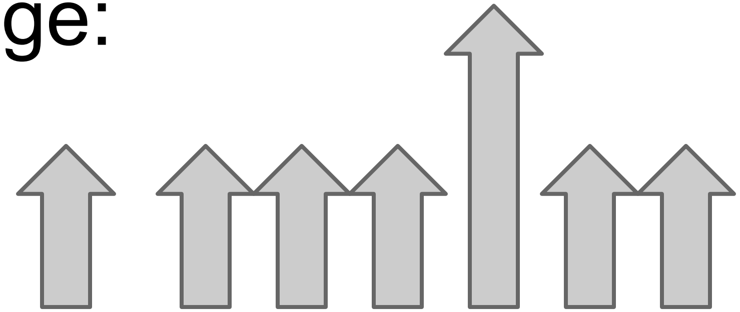
NP-hardness NP-completeness



Optimality of Grover search

Flipping around average:





Optimality: $O(N^{1/2})$: Gilles Brassard, Peter Høyer, Lov Grover

Mixing measurements and unitary transformations -naive approach

Measurement confines the state to a subspace, following the structure of a problem

Algorithm:

If |yes> done;

else backtrack //with unitary operator:H_{yes}*-> H*

BACKTRACK ?!

Simple U_S(as a matrix in computational basis in 6 dimensions, sorry Slides :-()

a	b	b	b	b	b
b	a	b	b	b	b
b	b	a	b	b	b
b	b	b	a	b	b
b	b	b	b	a	b
b	b	b	b	b	a

from unitarity: $a^2+(N-1)*b^2=1$, $2ab+(N-2)b^2=0$ => $a= 2/N -1$, $b=2/N$

Well, we have seen this before

$$U_s = 2\left|s\right\rangle\left\langle s\right| - I$$

==	(2/N -1)	2/N	…	…	...
	2/N	(2/N-1)	2/N	…	...
	2/N	2/N	(2/N-1)	2/N	…
	…	…	…	…	...

represents (again) Grover diffusion operator

Hamiltonian Circuit Problem

NP- Complete

- Bondy–Chvátal theorem says when a graph has a HC.
- Let’s create a sequence of graphs, starting from the complete graph, ending in a given graph with the property that if the following graph in the sequence has a HC then a previous one has one, too - stepping through the sequence can be done by measurements of a function testing Hamiltonicity of a current graph in a sequence.

Naive approach

- There is a structure in a HC problem, allowing for a construction of a sequence of graphs starting from the complete graph and ending with a graph for which a solution to HC problem is sought, where stepping through the sequence can be done by efficient (probability polynomial in the number of vertices) measurement
- Algorithmically useful backtracking is possible when a state is reduced to the “bad” subset of HCs (using Grover diffusion operator acting on appropriately chosen subspace).

Naive approach - continued

If a vertex a has an edge to all other edges in a given graph in the above construction, the Hamiltonian Cycles can be grouped into sets of n-1 or 1 (or 0), differing only in the position of n-th vertex in the same n-1 vertex cycle.

Backtracking revisited

After unsuccessful measurement i.e. one collapsing the state to a superposition of HCs of the previous graph in a sequence and NOT being HCs of the current one, backtracking can be done within above defined sets*

Theorem. *Let $c \in \mathbb{N}$, $0 \le t_j \le s_j$, $j = 1, \dots, c$. Let $j_0 \in \{1, \dots, c\}$ be such that with $0 < t_{j_0} = s_{j_0}$. Consider the state*

$$|S\rangle := \left(\sum_{j=1}^{c} s_j \right)^{-1/2} \sum_{j=1}^{c} \left(\sqrt{s_j - t_j} \, |j, 0\rangle + \sqrt{t_j} \, |j, 1\rangle \right) .$$

Assume that we are only allowed to transfer amplitude within each of the spaces U_j spanned by $\{|j,0\rangle, |j,1\rangle\}$, $j = 1, \dots, c$. Define

$$|T\rangle := \left(\sum_{j=1}^{c} t_j \right)^{-1/2} \sqrt{t_j} \, |j, 1\rangle .$$

Then, no sequence of operations from $|S\rangle$ to $|T\rangle$ yields higher success probability than a direct measurement M of the rightmost qubit.

Proof. We proceed by induction on the number k of measurements.

For $k = 0$ either $s_j = t_j$ for all $j = 1, \dots, c$ and we are done (the probability of success is 1) or $s_j < t_j$ for some $j = 1, \dots, c$ and there is no way of getting to $|T\rangle$ (the probability of success is 0).

Let $k \ge 1$ and let M' be the first measurement. Fix one of the possible results $|T'\rangle$ of M'. Let t'_j be the squared amplitude of $|T'\rangle$ in U_j, $j = 1, \dots, c$.

Define

$$\lambda_1 := \min\{t_j^{-1} t'_j : j = 1, \dots, c, \ t_j > 0\},$$
$$\lambda_2 := \min\{t_j^{-1} (s_j - t'_j) : j = 1, \dots, c, \ t_j > 0\}.$$

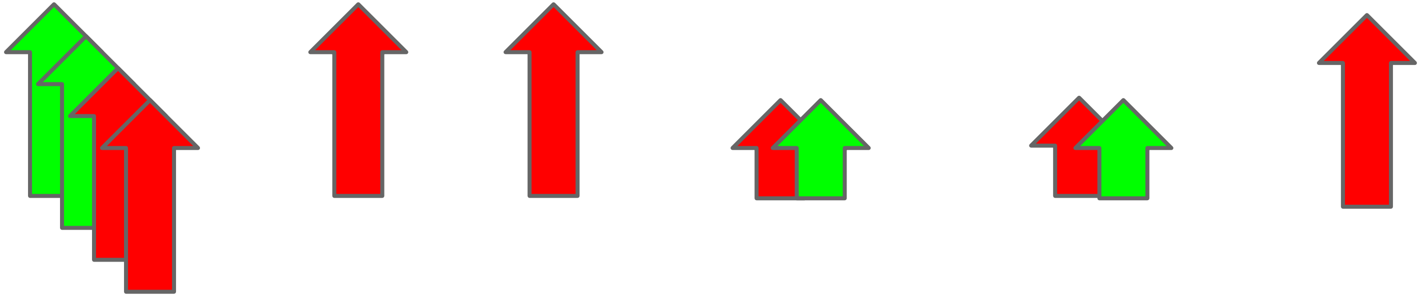
Observe that $\lambda_1 + \lambda_2 \le 1$. Let $j_1, j_2 \in \{1, \dots, c\}$ be such that $\lambda_1 = t_{j_1}^{-1} t'_{j_1}$ and $\lambda_2 = t_{j_2}^{-1}(s_{j_1} - t'_{j_1})$.

Applying the inductive hypothesis to $((s_j), j_0)$ replaced by $((t'_j/\lambda_1), j_1)$ and $(((s_j - t'_j)/\lambda_2), j_2)$, respectively, we may (in each of the cases) replace the remaining operations by a single measurement towards $|T\rangle$ with no loss of success probability.

The overall success probability after both measurements is

$$\left(\frac{\sum_{j=1}^{c} t'_j}{\sum_{j=1}^{c} s_j} \right) \left(\frac{\sum_{j=1}^{c} t_j}{\sum_{j=1}^{c} t'_j/\lambda_1} \right) + \left(\frac{\sum_{j=1}^{c} (s_j - t'_j)}{\sum_{j=1}^{c} s_j} \right) \left(\frac{\sum_{j=1}^{c} t_j}{\sum_{j=1}^{c} (s_j - t'_j)/\lambda_2} \right) = (\lambda_1 + \lambda_2) \frac{\sum_{j=1}^{c} t_j}{\sum_{j=1}^{c} s_j} \le \frac{\sum_{j=1}^{c} t_j}{\sum_{j=1}^{c} s_j}.$$

Inter-cell amplitude transfer

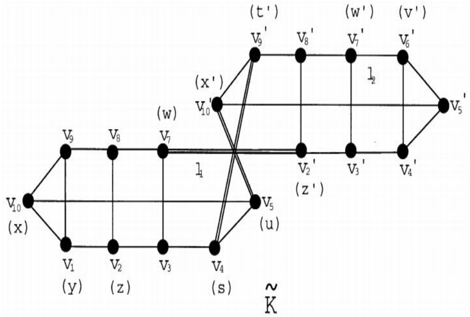


Naive efficient I-C amplitude transfer is tantamount to efficient classical algorithm

Inefficient existence proofs & optimal classical algorithms

Thomason's proof on Krawczyk' graph is exponential

Cameron, K. Discrete Mathematics 235 (2001) 69–77



Classical Eppstein algorithm is optimal

arXiv:cs/0302030v2

Combining of the two:

Sanity check: no, this is not a solution to Hamiltonian Path problem

Extensive simulations did not find paths of the auxiliary graphs (Thomason’s construction) longer than $O(n^2)$

Easy and hard cases