Distinguishability of quantum measurements

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Outline

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Quantum measurements

Definition of a measurement

A measurement (POVM) $\mathcal{M}$ on a complex Euclidean space X is a collection of non-negative operations $\{M_i\}_{i\in\Gamma}$ such that

$$\sum_{i\in\Gamma} M_i = \mathbb{1}_X. \tag{1}$$

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Application of a measurement

When a measurement is applied to a register whose state is ρ, an element of Γ is randomly selected as an outcome, with distribution

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Measurement application as quantum channel

An application of a measurement can be identified with a quantum–to–classical channel, defined as

$$\mathcal{M} : \rho \mapsto \sum_{i\in\Gamma} \mathrm{tr} M_i \rho |i\rangle\langle i|. \tag{3}$$

Quantum measurements

Orthogonal measurements

We are interested in projective rank-one measurements with given channel P_U, for which $M_i = U|i\rangle\langle i|U^\dagger = |u_i\rangle\langle u_i|$ and U is a unitary matrix.

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Discrimination of measurements

We know that a measurement $\mathcal{X}$ is one of two measurements: $\mathcal{M}$ or $\mathcal{N}$. We can prepare any input state ρ, and perform an unknown measurement. Using the output state, we want to decide whether $\mathcal{X} = \mathcal{M}$ or $\mathcal{X} = \mathcal{N}$.

We are working under an assumption that we have no knowledge on which of the measurements will be chosen, which can be written as $P(\mathcal{M}) = P(\mathcal{N}) = \frac{1}{2}$.

Classical approach

Scenario

The simplest idea is to distinguish random variables based on their probability vectors obtained after performing the measurements on some state ρ.

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Probability of correct guessing

Let $\mathcal{P}_{\mathbf{1}}$ and $\mathcal{P}_U$ be two projective measurements such that $U \in \mathcal{U}_d$ for arbitrary d. It holds the probability of correct guess is

$$\begin{aligned} &\frac{1}{2} + \frac{1}{4} \max_{\rho} \| \operatorname{diag}(\mathcal{P}_U(\rho)) - \operatorname{diag}(\mathcal{P}_{\mathbf{1}}(\rho)) \|_1 \\ &= \frac{1}{2} + \frac{1}{2} \max_{\Delta \subseteq \{1,\ldots,d\}} \left\| \sum_{i \in \Delta} (|i\rangle\langle i| - |u_i\rangle\langle u_i|) \right\|_{\infty} \\ &\geq 1 - \frac{1}{2} \min_j |u_{j,j}|^2 \end{aligned} \tag{4}$$

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Entangled input states

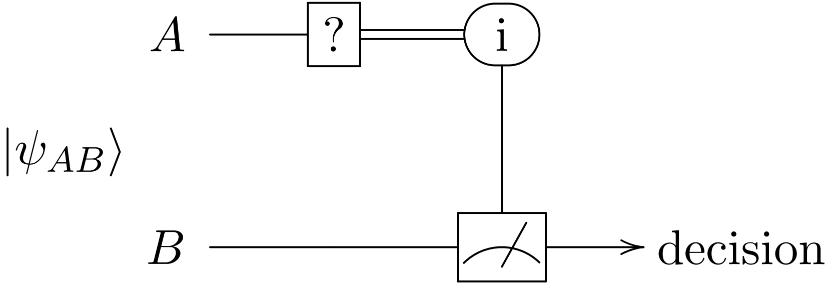

Figure: Entanglement assisted discrimination

Entanglement assisted discrimination

Diamond norm

Consider a linear mapping $\Phi : M_{d_1} \to M_{d_2}$. We define its completely bounded trace norm, also known as the diamond norm as

$$\|\Phi\|_\diamond = \max_{\|X\|_1=1} \|(\Phi \otimes \mathbb{1})(X)\|_1. \tag{5}$$

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Jamiołkowski-Choi isomorphism

For a given linear mapping $\Phi : M_{d_1} \to M_{d_2}$, we consider its Jamiołkowski-Choi map defined as

$$J(\Phi) = \left(\Phi \otimes \mathbb{1}\right)(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|). \tag{6}$$

Holevo-Helstrom theorem

For given states ρ_0, ρ_1, it holds that for every choice of binary measurement $\mu = \{\mu(0), \mu(1)\}$, we have that the probability p of correct discrimination satisfies

$$p = \frac{1}{2}\mathrm{tr}\mu(0)\rho_0 + \frac{1}{2}\mathrm{tr}\mu(1)\rho_1 \leq \frac{1}{2} + \frac{1}{4}\|\rho_0 - \rho_1\|_1. \tag{7}$$

Moreover, there exists a measurement for which the bound is saturated.

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Corollary

For given channels Φ_0, Φ_1, it holds that the optimal probability p of correct discrimination between channels is

$$p = \frac{1}{2} + \frac{1}{4}\|\Phi_0 - \Phi_1\|_\diamond. \tag{8}$$

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Useful property

$$\|\Phi\|_\diamond = \max\{\|(\mathbb{1} \otimes \sqrt{\rho})J(\Phi)(\mathbb{1} \otimes \sqrt{\rho})\|_1 : \rho \in \Omega_{d_1}\} \tag{9}$$

It is possible to obtain the value of the diamond norm on a channel extended by a $\mathrm{rank}(\rho)$-dimensional identity channel.

Distinguishability of unitary channels

For a given matrix A, we denote by $W(A)$ its numerical range i.e.

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Let $U \in \mathcal{U}_d$ and $\Phi_U : \rho \mapsto U\rho U^\dagger$ be a unitary channel. Then

$$\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2}, \tag{10}$$

where $\nu = \min\left\{|x| : x \in W(U^\dagger)\right\}$.

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$$
\begin{aligned}
\|\mathcal{P}_{\mathbf{1}} - \mathcal{P}_U\|_\diamond &= \max_\rho \left\|(\mathbb{1} \otimes \sqrt{\rho}) J(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_U)(\mathbb{1} \otimes \sqrt{\rho})\right\|_1 \\
&= \max_\rho \sum_{i=1}^{d} \sqrt{\left(\langle i|\rho|i\rangle + \langle u_i|\rho|u_i\rangle\right)^2 - 4\left|\langle i|\rho|u_i\rangle\right|^2}
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Proposition

Let $U \in \mathcal{U}_d$. Then $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ are perfectly distinguishable if and only if there exists $\rho \in \Omega_d$ such that

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Corollary

Measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ are perfectly distinguishable without use of an entanglement if and only if there exists rank-deficient principal submatrix of U.

Program to find if perfectly distinguishable

$$
\begin{aligned}
A_0 &= \mathbb{1} \\
A_i &= U|i\rangle\langle i| + |i\rangle\langle i|U^\dagger, \text{ for } i = 1, \ldots, d \\
A_{d+i} &= \mathrm{i}\left(|i\rangle\langle i|U^\dagger - U|i\rangle\langle i|\right), \text{ for } i = 1, \ldots, d.
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Hence we arrive at the primal and dual problems

Primal problem

maximize: $\operatorname{tr}\rho A_0$

subject to: $\operatorname{tr}\rho A_i = 0$

$\operatorname{tr}\rho = 1$

$\rho \in \mathcal{H}_d^+$

Dual problem

minimize: $\langle 0|Y|0\rangle$

subject to: $\sum_{i=0}^{2d} A_i Y_{ii} \geq \mathbb{1}$

$Y \in \mathcal{H}_d.$

Approximation method

1. $\rho \leftarrow \frac{1}{d}\mathbb{1}$
2. $|\rho_i\rangle \leftarrow |\rho_i\rangle - \langle u_i||\rho_i\rangle|u_i\rangle$ for $i = 1, \ldots, d$
3. $\rho \leftarrow \frac{\sqrt{\rho\rho^\dagger}}{\mathrm{tr}\sqrt{\rho\rho^\dagger}}$
4. Repeat $2, 3$

The main result

Theorem

Let $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ be two projective measurements and $\mathcal{DU}$ be the set of diagonal unitary matrices. Then

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \min_{E \in \mathcal{DU}} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \tag{14}$$

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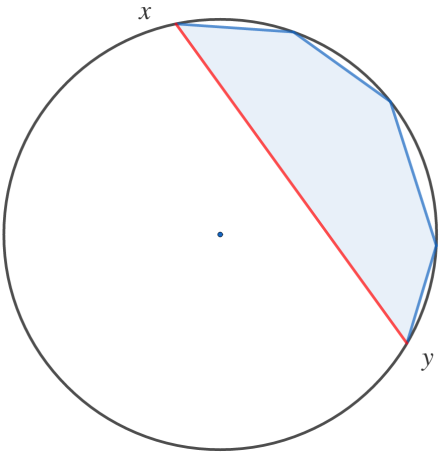
Corollary

Let $U \in \mathcal{U}_d$. Then $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ are perfectly distinguishable if and only if for all $E \in \mathcal{DU}$, unitary channel Φ_{UE} is perfectly distinguishable from the identity channel $\Phi_{\mathbf{1}}$.

The main result

$$\min_{E \in \mathcal{DU}} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1 - \max_{E \in \mathcal{DU}} \min_{\rho \in \Omega_d} |\mathrm{tr} \rho UE|^2}. \tag{15}$$



Examples

Necessary and sufficient conditions for perfect discrimination

Let $U \in \mathcal{U}_d$ and $E \in \mathcal{DU}$ be such that $\langle i|UE|i\rangle \geq 0$. Then the following holds:

- if P_U and $P_{\mathbf{1}}$ are perfectly distinguishable, then $\mathrm{tr}(UE) \leq d-2$,
- if $\mathrm{tr}(UE) \leq 1$, then P_U and $P_{\mathbf{1}}$ are perfectly distinguishable for odd $d \geq 3$.

In particular, if $d=3$, then P_U and $P_{\mathbf{1}}$ are perfectly distinguishable if and only if $\mathrm{tr}(UE) \leq 1$.

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In particular, if $d = 3$, then P_U and $P_{\mathbf{1}}$ are perfectly distinguishable if and only if $\mathrm{tr}(UE) \leq 1$.

Let $U \in \mathcal{U}_{2d}$ be such that $\lambda(U) = \{\lambda, \overline{\lambda}\}$, $\Re(\lambda) > 0$, $\dim(\Pi_\lambda) = \dim(\Pi_{\overline{\lambda}})$ and its eigenspace is given by an arbitrary complex Hadamard unitary matrix V, that is $|V_{i,j}| = \frac{1}{\sqrt{2d}}$. Hence, if $\lambda \to \mathrm{i}$ we obtain $|U_{i,i}(\lambda)| = \Re(\lambda) \to 0$, but $\|\mathcal{P}_{U(\lambda)} - \mathcal{P}_{\mathbf{1}}\|_\diamond \nearrow 2$.

Examples

Consider two binary measurements $\mathcal{M} = \{M_1, \mathbb{1} - M_1\}$ and $\mathcal{N} = \{N_1, \mathbb{1} - N_1\}$. Then

$$\|\mathcal{M} - \mathcal{N}\|_\diamond = 2\|M_1 - N_1\|_\infty. \tag{16}$$

Furthermore, we do not need entanglement to achieve this value.

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Fourier matrix

Let $d \geq 4$, then a measurement in the Fourier basis $\mathcal{P}_{F_d}$ is perfectly distingushable from $\mathcal{P}_{\mathbf{1}}$ by using a discriminator ρ, such that $\mathrm{rank}(\rho) \leq 2$.
Moreover, if d is not a square-free number, then the discrimination can be done without the use of an entanglement.

Examples

Reflection matrix

Let $U = \mathbb{1} - 2|x\rangle\langle x|$. $\mathcal{P}_U$ is perfectly distinguishable from $\mathcal{P}_{\mathbf{1}}$ if and only if $\omega = \max_i |x_i|^2 \le \frac{1}{2}$. Optimal discriminator ρ has $\mathrm{rank}(\rho) \le 2$.
Moreover, we can distinguish the above measurements without use of an entanglement if and only if

$$\exists_{\Gamma \subset \{0,1,\ldots,d-1\}} : \sum_{i \in \Gamma} |x_i|^2 = \frac{1}{2}. \tag{17}$$

In the case when $\omega > \frac{1}{2}$, we have $\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = 2\sqrt{1 - 4(\omega - \frac{1}{2})^2}$.

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Open question

Consider two unitary matrices U, V, such that they are perfectly distinguishable from the identity matrix $\mathbb{1}$, $U = C \operatorname{diag}^\dagger(\lambda) C^\dagger$ and $V = D \operatorname{diag}^\dagger(\lambda) D^\dagger$. If for arbitrary $p \in [0,1]$ and given bistochastic matrices $B(C), B(D)$ the matrix $E = pB(C) + (1-p)B(D)$ is unistochastic, then the matrix $B^{-1}(E) \operatorname{diag}^\dagger(\lambda) B^{-1}(E)^\dagger$ is perfectly distinguishable from $\mathbb{1}$.

Sketch of proof

Theorem

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \max_\rho \left\|(\mathbb{1} \otimes \sqrt{\rho}) J(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_U)(\mathbb{1} \otimes \sqrt{\rho})\right\|_1 = \min_{E \in \mathcal{DU}} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \quad (18)$$

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Proof

- $$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond \leq \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \quad (19)$$

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$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \max_{\rho} \|(\mathbb{1} \otimes \sqrt{\rho}) J(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_U)(\mathbb{1} \otimes \sqrt{\rho})\|_1 = \min_{E \in \mathcal{D}\mathcal{U}} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \quad (18)$$

Proof

- $$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond \leq \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \quad (19)$$
- $$\min_{\rho \in \Omega_d} \max_{E \in \mathcal{D}_{\leq 1}} \Re\left(\mathrm{tr}(\rho U E)\right) = \max_{E \in \mathcal{D}_{\leq 1}} \min_{\rho \in \Omega_d} \Re\left(\mathrm{tr}(\rho U E)\right). \quad (20)$$

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- $$\min_{\rho \in \Omega_d} \max_{E \in \mathcal{DU}} |\mathrm{tr}(\rho U E)| = \max_{E \in \mathcal{DU}} \min_{\rho \in \Omega_d} |\mathrm{tr}(\rho U E)|. \quad (21)$$

Convex program for calculating diamond norm

Convex problem

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\begin{aligned}
\text{minimize:} \quad & \sum_i |\langle i|\rho U|i\rangle| \\
\text{subject to:} \quad & \mathrm{tr}\rho = 1, \\
& \rho \geq 0.
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$$\begin{aligned} \min_{E\in\mathcal{DU}} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond &= 2\sqrt{1 - \max_{E\in\mathcal{DU}} \min_{\rho\in\Omega_d} |\mathrm{tr}\rho UE|^2}. \\ &= 2\sqrt{1 - \left(\min_{\rho\in\Omega_d} \sum_i |\langle i|\rho U|i\rangle|\right)^2}. \end{aligned} \tag{23}$$

Sketch of proof

Proof

- If $0 < \min_{\rho \in \Omega_d} |\mathrm{tr} \rho U E_0| = \max_{E \in \mathcal{DU}} \min_{\rho \in \Omega_d} |\mathrm{tr} \rho U E|$, then the optimality condition forces that there exits a state $\rho_0 = \frac{1}{2}(\rho_x + \rho_y)$ that minimizes $|\mathrm{tr} \rho U E_0|$ and $(\rho_x)_{i,i} = (\rho_y)_{i,i}$ for the most distant pair of eigenvalues x, y.

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- Given that the state ρ_0 is the discriminator

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\begin{aligned}
2\sqrt{1 - \left|\frac{x+y}{2}\right|^2} &= \|(\mathbb{1} \otimes \sqrt{\rho_0}) J(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_U)(\mathbb{1} \otimes \sqrt{\rho_0})\|_1 \\
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- The case when $0 \in W(UE)$ for all $E \in \mathcal{DU}$ follows from SDP program to find if perfectly distinguishable. The primal problem has a solution $\rho \geq 0$ if and only if for all real vectors $(x_1, \ldots, x_{2d}) \in \mathbb{R}^{2d}$ we have $0 \in W\left(\sum_{i=1}^{2d} x_i A_i\right)$.

Perturbation of numerical range of a unitary matrix

Problem

Let $U \in \mathcal{U}_d$. Our aim is to calculate $\max_{E \in \mathcal{DU}} \min_{\rho \in \Omega_d} |\mathrm{tr} \rho U E|$.

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$$E(t) = \mathrm{diag}^{\dagger}(\{e^{\mathrm{i} p_i t}\}).$$

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Behavior of eigenvalues

Consider that the matrix U has a non-degenerate eigenvalue λ with eigenvector $|x\rangle$, then for small enough time $t > 0$

$$\lambda_t(UE(t)) \approx \lambda e^{\mathrm{i} t \sum |x_i|^2 p_i} \tag{25}$$

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$$\lambda_t(UE(t)) \approx \lambda e^{\mathrm{i} t \sum |x_i|^2 p_i} \tag{25}$$

Let $S_\lambda = \{|x\rangle : (\lambda \mathbb{1} - U)|x\rangle = 0, \|x\| = 1\}$. Then for small enough time $t > 0$

$$\begin{aligned} \lambda_t(UE(t))_{slowest} &\approx \lambda \exp\left(\mathrm{i} t \min_{|x\rangle \in S_\lambda} \sum |x_i|^2 p_i\right) \\ \lambda_t(UE(t))_{fastest} &\approx \lambda \exp\left(\mathrm{i} t \max_{|x\rangle \in S_\lambda} \sum |x_i|^2 p_i\right). \end{aligned} \tag{26}$$

Perturbation of numerical range of a unitary matrix

Corollary

Assume that x, y are the most distant eigenvalues of U. Then

1.
$$|x - y| = \max_{E \in \mathcal{DU}} |x(E) - y(E)| \iff \exists_{\rho_x, \rho_y} \rho_x = \Pi_x \rho_x \Pi_x, \rho_y = \Pi_y \rho_y \Pi_y, \quad (\rho_x)_{i,i} = (\rho_y)_{i,i}. \tag{27}$$

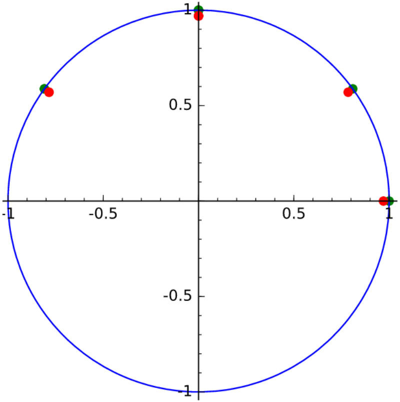
2.
$$|x - y| < \max_{E \in \mathcal{DU}} |x(E) - y(E)| \iff \exists_p \min_{|x\rangle \in S_x} \sum |x_i|^2 p_i > \max_{|y\rangle \in S_y} \sum |y_i|^2 p_i. \tag{28}$$

Example

0.061185	0.010173	0.217431	0.003623	0.128876	0.22385	0.140743
0.368343	0.143865	0.002198	0.151242	0.149181	0.047676	0.112829
0.103951	0.063953	0.190502	0.087191	0.025167	0.457315	0.179746
0.061299	0.06592	0.219399	0.366221	0.436046	0.01896	0.12005
0.038088	0.377464	0.135186	0.125733	0.027001	0.011093	0.209557
0.057563	0.233291	0.083226	0.020552	0.193159	0.181138	0.203362
0.309571	0.105333	0.152058	0.245438	0.040569	0.059969	0.033712

h1 0.0
h2 0.0
h3 0.0
h4 0.0
h5 0.0
h6 0.0
h7 0.0

Dist=0.309016994374947

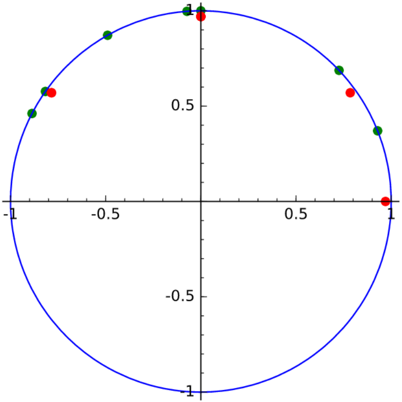


Example

0.098177	0.004139	0.004271	0.4345	0.004559	0.428187	0.026167
0.262093	0.137336	0.0	0.082124	0.310873	0.015663	0.191911
0.13111	0.057108	0.169137	0.150098	0.019024	0.092573	0.380949
0.098003	0.044375	0.722124	0.012042	0.000307	0.020882	0.102266
0.035774	0.496579	0.00279	0.118582	0.105659	0.168418	0.072197
0.129051	0.197734	0.101678	0.151252	0.046196	0.244943	0.129145
0.245791	0.062729	0.0	0.051401	0.513381	0.029334	0.097365

h1 0.0
h2 0.1
h3 0.0
h4 0.0
h5 0.0
h6 0.0
h7 0.1

Dist=0.416516074166020

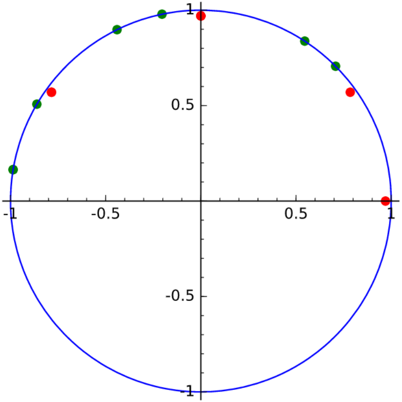


Example

$$\begin{pmatrix} 0.072979 & 0.018666 & 0.017827 & 0.23461 & 0.033704 & 0.515391 & 0.106823 \\ 0.148946 & 0.190325 & 0.003555 & 0.070998 & 0.292279 & 0.119016 & 0.17488 \\ 0.184318 & 0.070076 & 0.263642 & 0.153792 & 0.037533 & 0.148842 & 0.141797 \\ 0.118348 & 0.014316 & 0.673985 & 0.064627 & 0.006691 & 0.071343 & 0.050691 \\ 0.097774 & 0.53877 & 0.004119 & 0.110205 & 0.120507 & 0.089167 & 0.039458 \\ 0.214136 & 0.079835 & 0.035559 & 0.301695 & 0.087349 & 0.040829 & 0.240597 \\ 0.1635 & 0.088012 & 0.001312 & 0.064074 & 0.421937 & 0.015412 & 0.245753 \end{pmatrix}$$

h1	0.1
h2	0.1965
h3	0.0003
h4	0.0401
h5	0.011
h6	0.052
h7	0.201

Dist=0.457114728695838



Thank you for your attention!