Uncertainty relations from numerical range

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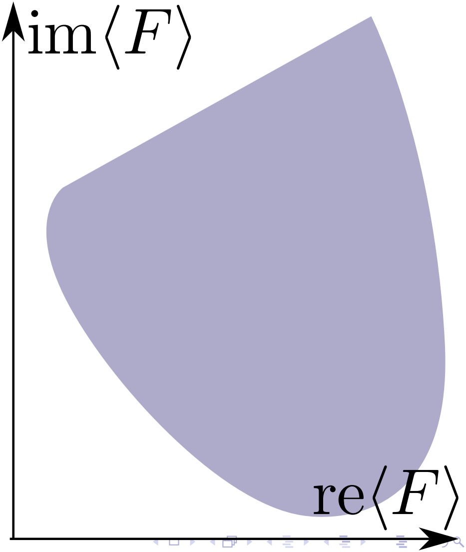
Numerical range

Numerical range of operator F acting on d-dimensional space is

$$W(F) = \{\langle\psi|F|\psi\rangle : |\psi\rangle \in \mathbb{C}^d, \langle\psi|\psi\rangle = 1\},$$

- $W(F)$ is convex and compact (Hausdorff and Toeplitz, 1919),



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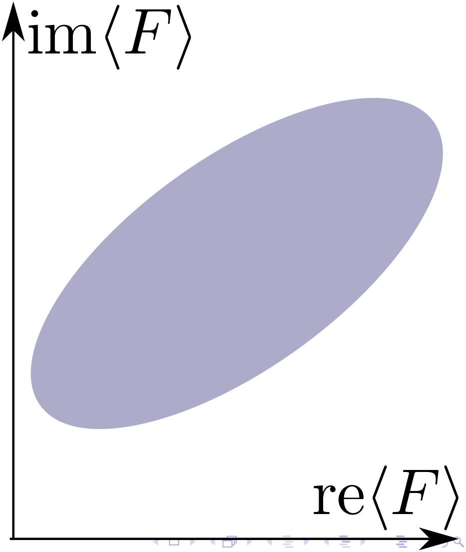
- if $d = 2$, $W(F)$ is an ellipse,



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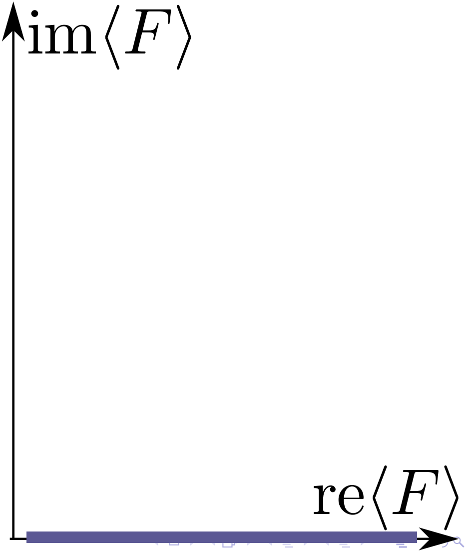
- $W(F)$ is convex and compact (Hausdorff and Toeplitz, 1919),
- if F is Hermitian, $W(F)$ is a segment on real line from minimal to maximal eigenvalue $[\min\lambda(F), \max\lambda(F)]$,



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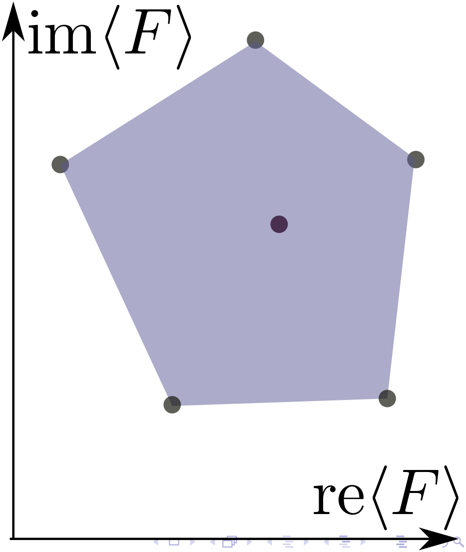
- $W(F)$ is convex and compact (Hausdorff and Toeplitz, 1919),
- if hermitian part of F, $F_H = (F + F^\dagger)/2$, commutes with antihermitian $F_A = (F - F^\dagger)/2i$, $W(F)$ is a polygon – convex hull of eigenvalues of F.

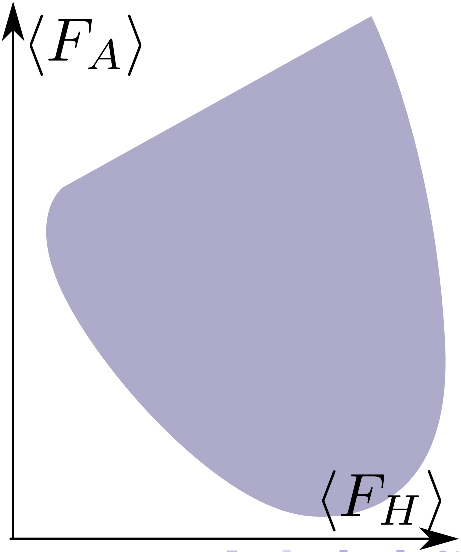


Joint numerical range

Joint numerical range is a set of simultaneously attainable averages of several hermitian operators $F_1, \ldots, F_k$:

$$W(F_1, \ldots, F_k) = \left\{ \left(\langle F_1 \rangle_{|\psi\rangle}, \ldots, \langle F_k \rangle_{|\psi\rangle} \right) : |\psi\rangle \in \mathbb{C}^d,\ \langle \psi | \psi \rangle = 1 \right\},$$

- Numerical range of single operator $W(F)$ is isomorphic to joint numerical range of its hermitian and antihermitian part, $W(F_H, F_A)$.



Uncertainty relations

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$$\Delta^2 A \Delta^2 B \geq |\frac{1}{2}\langle[A, B]\rangle|^2.$$

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$$\Delta^2 A + \Delta^2 B \geq \pm i\langle [A, B] \rangle + |\langle \psi | A + iB | \psi^{\perp} \rangle|^2$$

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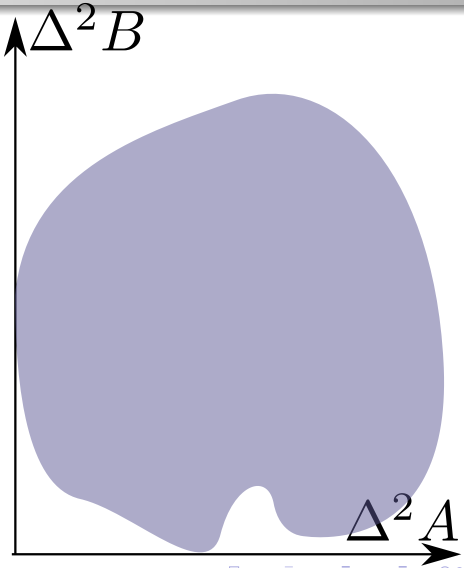
In general, all of them are contained in a single image, *uncertainty range*:



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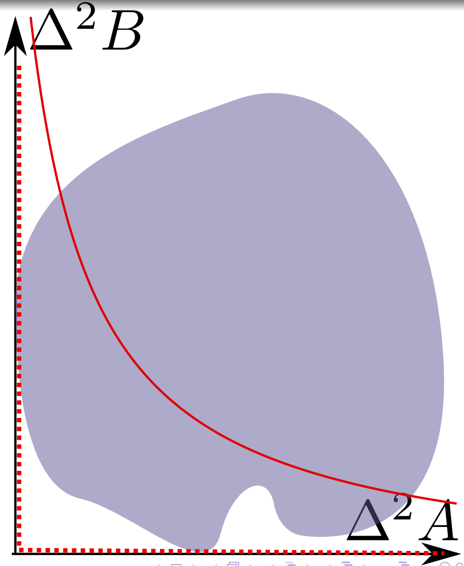

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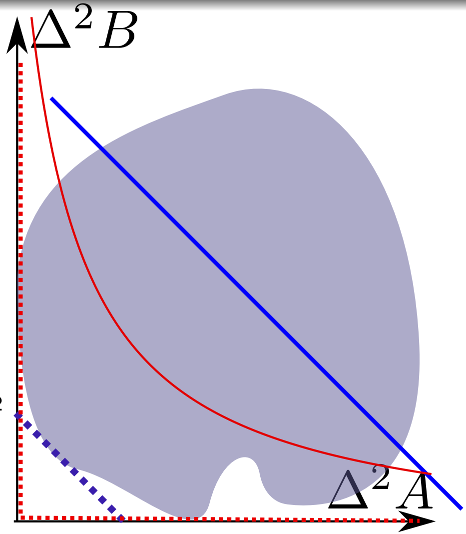


Geometrical approach to uncertainty relations

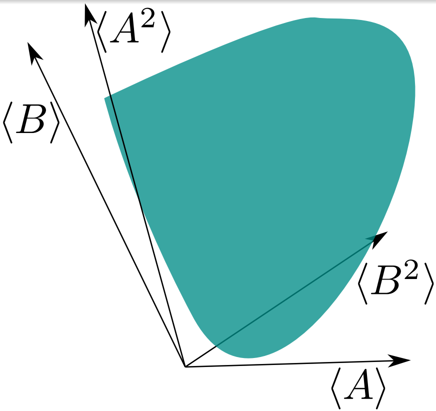
For fixed operators A, B consider

$$W(A, B, A^2, B^2)$$

For all points $(a, b, \bar{a}, \bar{b}) \in W(A, B, A^2, B^2)$ apply transformations

- $(a, b, \bar{a}, \bar{b}) \mapsto (a^2, b^2, \bar{a}, \bar{b})$,
- $(a^2, b^2, \bar{a}, \bar{b}) \mapsto (\bar{a} - a^2, \bar{b} - b^2)$.

In the end, we obtain *uncertainty range*:



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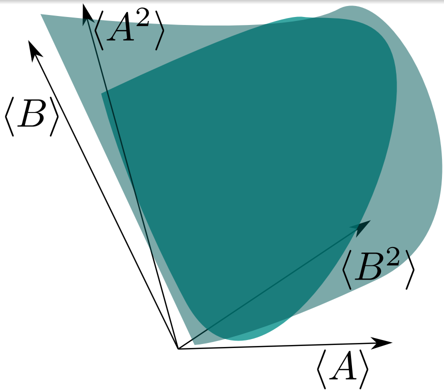

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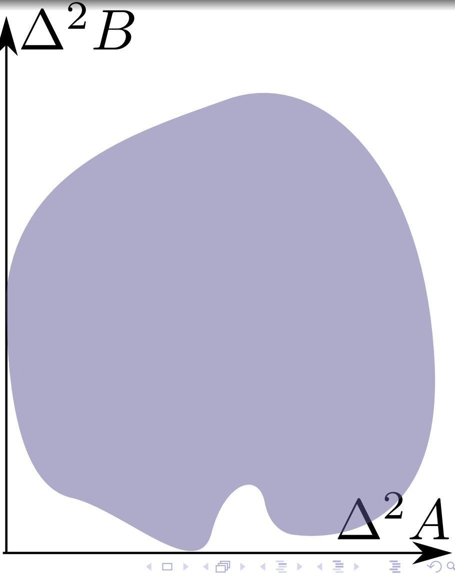
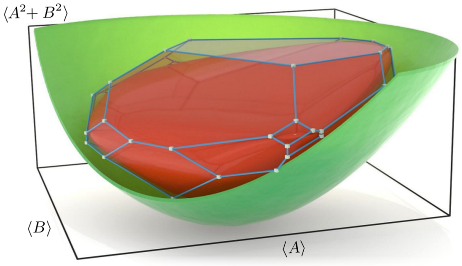


Maccone-Pati uncertainty relation

$$\Delta^2 A + \Delta^2 B = \langle A^2 + B^2 \rangle - \langle A \rangle^2 - \langle B \rangle^2$$

$W(A, B, A^2 + B^2)$ is sufficient to determine the bound.

- numerical procedure has been recently proposed (R. Schwonnek et al., PRL 2017; KS MSc thesis),
- analytical treatment is possible for low-dimensional cases.

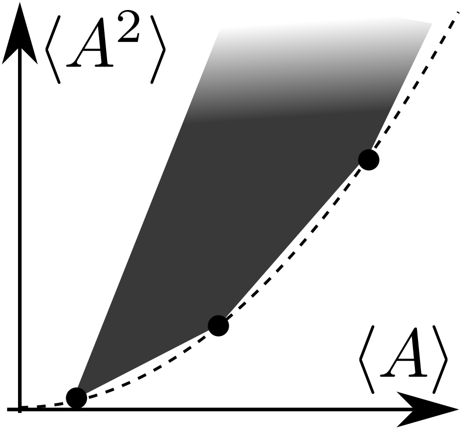


Source: R. Schwonnek et al., PRL 2017

Approximations of uncertainty range

Variance $\Delta^2 A$ can be approximated the following way:

- calculate eigenvalues of A, $\lambda(A)$ ($W(A, A^2)$ is a polygon with vertices at $(\lambda, \lambda^2), \quad \lambda \in \lambda(A)$),



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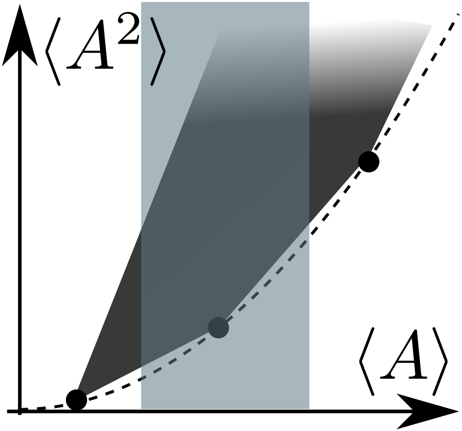

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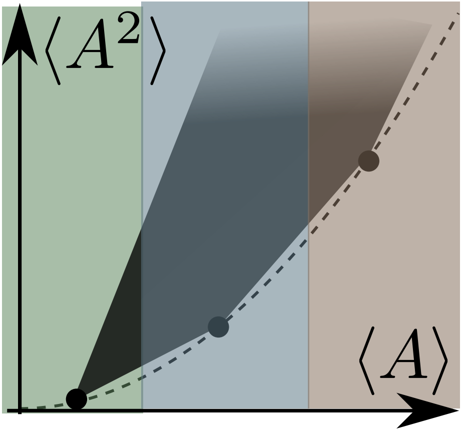
- proceed for all vertices – obtain complete set of approximations $\{\bar{A}_i\}$



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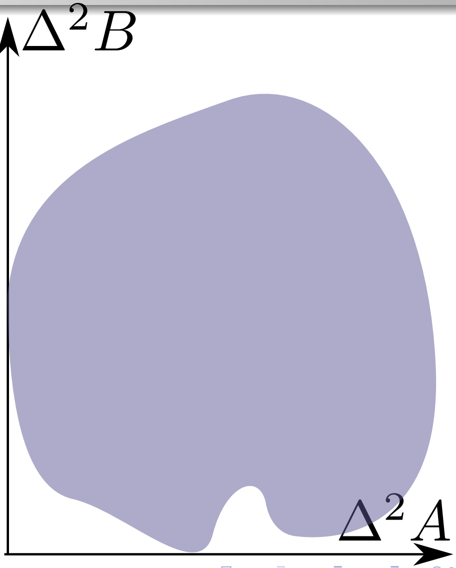
From sets of approximations of two operators $\{\bar{A}_i\}$, $\{\bar{B}_j\}$ approximation of uncertainty range can be constructed:

- take any $\bar{A}_i, \bar{B}_j$,



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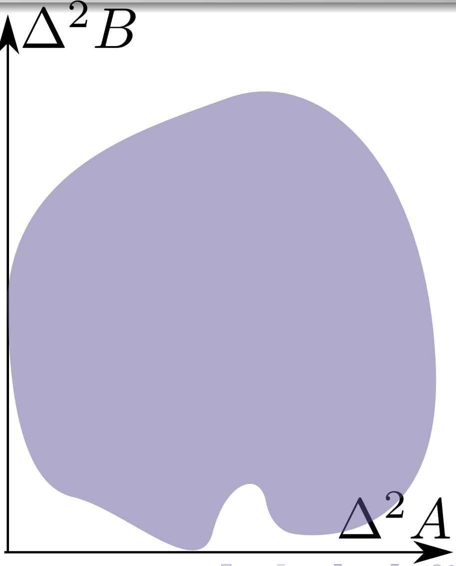
From sets of approximations of two operators $\{\bar{A}_i\}, \{\bar{B}_j\}$ approximation of uncertainty range can be constructed:

- take any $\bar{A}_i, \bar{B}_j$,
- calculate joint numerical range $W(\bar{A}_i, \bar{B}_j)$,



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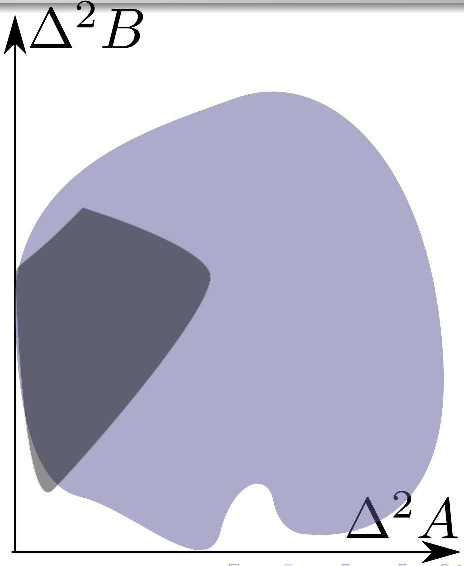

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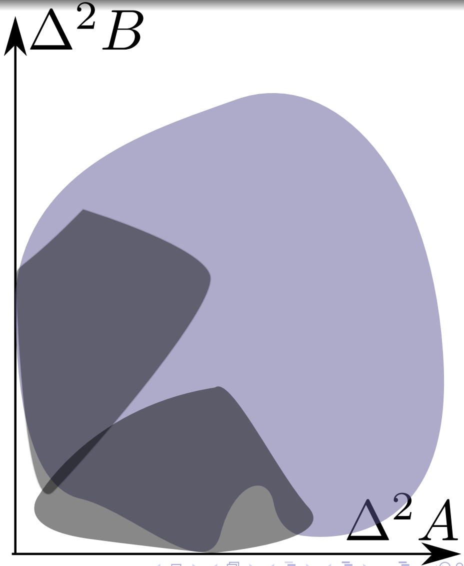
- complete approximation is a sum of all possible pairs $W(\bar{A}_i, \bar{B}_j)$.



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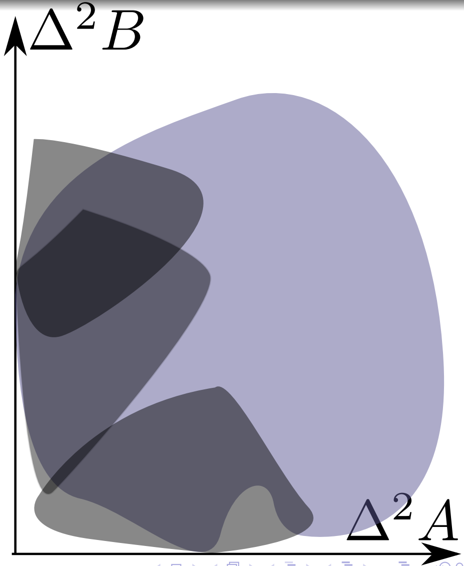
- resulting set is a part of the **lower** approximation of *uncertainty range*,
- $\Delta^2 A + \Delta^2 B \geq c(A, B)$, where $c = \min_{i,j}(\min \lambda(\bar{A}_i + \bar{B}_j))$.



Conclusions

- Numerical ranges are useful tools in studies of uncertainty relations,
- they provide a new geometrical look on the topic,
- new explicit bounds for the **sum** of variances are derived,
- semi-analitical methods provide **tight** bounds in low dimensions.

References I

L. Maccone, A.K. Pati, *Stronger Uncertainty Relations for the Sum of Variances*, PRL 2014

R. Schwonnek, L. Dammeier, R.F. Werner, *State-independent Uncertainty Relations and Entanglement Detection in Noisy Systems*, PRL 2017

K. Szymański, *Uncertainty relations and joint numerical ranges*, arXiv 1707.03464