Tight t-designs

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1. Frame theory

2. t-designs
   - Complex projective t-designs
   - Tight t-designs

3. Conclusion

Definition

Let $\mathcal{H}$ be a finite dimensional vector space. Then we call a collection of vectors $|\phi_k\rangle \in \mathcal{H}$ a frame if there exist contants $0 < a \leqslant b < \infty$ such that

$$a|\langle\xi|\xi\rangle|^2 \leqslant \sum_k |\langle\xi|\phi_k\rangle|^2 \leqslant b|\langle\xi|\xi\rangle|^2$$

for all $|\xi\rangle \in \mathcal{H}$. The contants a and b are called the frame bounds. If $a = b$, then the frame is said to be tight.

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Remark (POVM -> frame)

Let $(\Omega, \mathcal{A})$ such that $|\Omega| < \infty$ be a measurable space and let $\mathcal{L}_s^+(\mathcal{H})$ be the set of self-adjoint positive-semidefinite operators on $\mathcal{H}$. Then the (finite) positive operator-valued measure (POVM) is defined to be a set of operators $\{\Pi_j\}_{j\in\Omega} \subset \mathcal{L}_s^+(\mathcal{H})$ satisfying the identity decomposition:

$$\sum_{j\in\Omega} \Pi_j = I$$

Special class of such POVMs are normalized rank-one POVMs where Π_j $(j = 1, \ldots, n)$ are rank-one operators and $Tr(\Pi_j) = \frac{d}{n}$, where d is the dimension of $\mathcal{H}$.

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Remark

For a tight frame the positive operator $S = \sum_k |\phi_k\rangle\langle\phi_k| = aI$. *This tight frame condition is equivalent to the completeness condition for the corresponding POVM elements* $|\phi_k\rangle\langle\phi_k|/a$. *Thus: rank-one POVMs and tight frames are the same mathematical object.*

Definition

Let $S_d = \{v \in \mathbb{C}^d : \|v\| = 1\}$, $\{|\phi_k\rangle\}_{k=1}^n \subset S_d$. *We will define frame potential as*

$$Tr[S^2] = \sum_{j,k} |\langle\phi_j|\phi_k\rangle|^2$$

Theorem (Benedetto-Fickus)

Given any d and n, let $\{|\phi_k\rangle\}_{k=1}^{n} \subset S_d$ be a set of normalized vectors with frame operator S. Then:

$$Tr[S^2] \geqslant \max(n, n^2/d)$$

Furthermore, the bound is achieved iff $\{|\phi_k\rangle\}$ consists of orthonormal vectors, when $n \leqslant d$, or is a tight frame, when $n \geqslant d$.

Remark

The frame potential of $P = \{|\phi_k\rangle\}_{k=1}^{n} \subset S_d$ is equal to $Tr[S^2] = d^3 = n^2/d$, so immediately we can deduce that P is a tight frame and also a POVM.

Remark

Let us remind that for POVM to be informationally complete (IC-POVM), the d^2 operators $\Pi_k = |\phi_k\rangle\langle\phi_k|$ must be linearly independent so that they span the space of operators (it follows from considering the rank of their Gram matrix).

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Definition

A complex projective t-design is a set of n pure states $\{\rho_k\}_{k=1}^n \subset P(\mathbb{C}^d)$ such that

$$\frac{1}{n^2}\sum_{k,l=1}^{n} f(tr(\rho_k\rho_l)) = \iint_{P(\mathbb{C}^d)^2} f(tr(\rho\sigma))d\mu_H(\rho)d\mu_H(\sigma)$$

for any t-th order polynomial f, where μ_H denotes the unique unitarily invariant probability measure on $P(\mathbb{C}^d)$.

Equivalently:

$$\frac{1}{n}\sum_{k=1}^{n}\rho_k^{\otimes t} = \binom{d+t-1}{t}^{-1} P_s$$

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Definition

$X \subset S_d$ is a s-distance set if

$$|\{|x^* y|^2 : x, y \in X, x \neq y\}| = s$$

Theorem (Delsarte, Goethals, Seidel (1975))

If $X \subset S_d$ is a s-distance set, then:

$$|X| \leqslant \binom{d+s+1}{s}^2,$$

with equality if and only if X is a $2s$-design. If X is a $2t$-design, then:

$$|X| \geqslant \binom{d+t+1}{t}^2$$

Remark

$X \subset S_d$ is a complex t-design if

$$\frac{1}{|X|}\sum_{v\in X}(vv^*)^{\otimes t} = \int_{S_d}(vv^*)^{\otimes t}dv$$

Theorem (Renes, Blume-Kohout, Scott, Caves, 2003)

For any finite $X \subset S_d$

$$\frac{1}{|X|^2}\sum_{u,v\in X}|u^*v|^{2t} \geqslant \binom{d+t-1}{t}^{-1}$$

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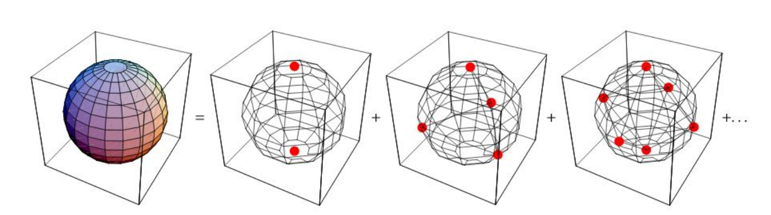
Example

In $\mathbb{C}^2$:

- *1-design: triangular bipyramid*
- *2-design: tetrahedron*
- *3-design: cube, octahedron*
- *5-design: icosahedron, dodecahedron*

Example



Example

Every rank-one normalized POVM can be considered as a complex projective 1-design.

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Every tight frame can be considered as a complex projective 1-design.

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Every orthonormal base can be considered as a complex projective 1-design.

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Example

A symmetric informationally complete POVM (SIC-POVM) consists of d^2 subnormalized rank-one projections: $\Pi_j = |\phi_j\rangle\langle\phi_j|/d$ such that:

$$tr(\Pi_i^* \Pi_j) = \frac{|\langle\phi_i|\phi_j\rangle|^2}{d^2} = \frac{1}{d^2(d+1)}$$

for $i \neq j$. The stated $\{|\phi_j\rangle\langle\phi_j|\}_{j=1}^{d^2}$ constitute a complex projective 2-design.

Example

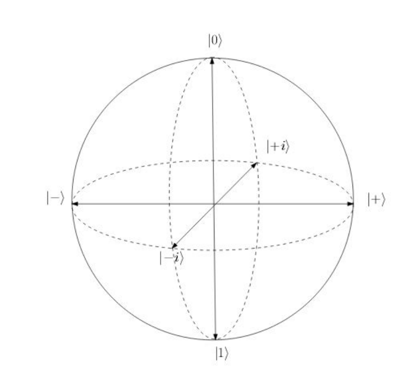
The complete set of mutually unbiased bases (MUB) is a set of $d+1$ orthonormal bases $\{e_i^j\}_{i=1,j=1}^{d,d+1}$ in $\mathbb{C}^d$ such that

$$|\left\langle e_i^j \middle| e_k^l \right\rangle|^2 = \frac{1}{d}$$

for $j \neq l$. The stated $\{\left|e_i^j\right\rangle \left\langle e_i^j\right|\}_{i=1,j=1}^{d,d+1}$ constitute a complex projective 2-design.

Example

A set of 3 MUBs in the Hilbert space of dimension 2 are 3-designs:



Definition

Let $\mathcal{B}$ be a finite s-distance set ($|B| = b$) in $P(\mathbb{C}^d)$ with A the set of values of $|\rangle x, y\langle|^2$ for all pairs $x, y \in \mathcal{B}$ and $e = |A \setminus \{0\}|$, $s = |A|$. Then we define special bound:

$$\frac{(d)_s(d)_e}{(1)_s e!}$$

where $(p)_a = p(p+1)\dots(p+a-1)$.

Theorem

Let $\mathcal{B}$ be a finite s-distance set ($|B| = b$) in $P(\mathbb{C}^d)$ with A the set of values of $|\rangle x, y\langle|^2$ for all pairs $x, y \in \mathcal{B}$ and $e = |A \setminus \{0\}|$, $s = |A|$. Let $\mathcal{B}$ be a t-design. Then: $t \leqslant s + e$, where equality hold iff b equals the absolute bound

Definition

Such t-designs for which the inequality in latter theorem is met are called tight t-designs.

Example

In $\mathbb{R}^2$ *for* $b = 4$ *we have vectors* $(1, 0), (0, 1), (1, 1), (1, -1)$.

Example

In $\mathbb{C}^2$ *for* $b = 6$ *we have vectors* $(1, 0), (0, 1), (1, 1), (1, -1), (1, i), (1, -i)$.

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In $\mathbb{C}^2$ *for* $b = 6$ *we have vectors* $(1,0),(0,1),(1,1),(1,-1),(1,i),(1,-i)$.

Example

For POVMs in $\mathbb{C}^d$ *we have* $s = e = 1$, *then the absolute bound is equal:* $\frac{d \cdot d}{1 \cdot 1} = d^2$ *and it is equal to* $b = d^2$. *That means that POVM are tight t-designs for every* d *and* $t = s + e = 2$.

Example

For $\mathbb{C}^d$ we have $s = 2, e = 1$, then the absolute bound is equal: $\frac{d \cdot d \cdot (d+1)}{2 \cdot 1} = \frac{d^2(d+1)}{2}$ and it is equal to $b = d(d+1)$ iff $d = 2$. That means that only for $d = 2$ POVMs are tight t-designs and $t = s + e = 3$.

Example

In $\mathbb{C}^4$ *for* $A = \{0, \frac{1}{3}\}$, $b = 40$ *we have diameters of the Witting polytope*

Vectors e_1, e_2, e_3, e_4 *and*
$(0, 1, -\omega_1, \omega_2), (-\omega_1, 0, 1, \omega_2), (\omega_1, -\omega_2, 0, 1), (1, \omega_1, \omega_2, 0)$; *where* ω_1, ω_2 *run independently through the cube roots of 1.*

Tight t-designs (absolute bound attained)						t-designs meeting special bound					
Example	$\mathbf{F}^n$	A	b	s	t	Example	$\mathbf{F}^n$	A	b	s	t
1	$\mathbf{R}^2$	$\{0, \frac{1}{2}\}$	4	2	3	12	$\mathbf{C}^3$	$\{0, \frac{1}{4}, \frac{1}{2}\}$	21	3	3
2	$\mathbf{C}^2$	$\{0, \frac{1}{2}\}$	6	2	3	13	$\mathbf{H}^4$	$\{0, \frac{1}{4}, \frac{1}{2}\}$	180	3	3
3	$\mathbf{H}^2$	$\{0, \frac{1}{2}\}$	10	2	3	14† n even (≥ 4)	$\mathbf{C}^n$	$\left\{\frac{1}{2n-1}\right\}$	$2n$	1	1
·4	$\mathbf{O}^2$	$\{0, \frac{1}{2}\}$	18	2	3	15† n odd (≥ 3)	$\mathbf{H}^n$	$\left\{\frac{1}{2n-1}\right\}$	$2n$	1	1
5	$\mathbf{C}^3$	$\{\frac{1}{4}\}$	9	1	2	16	$\mathbf{C}^3$	$\{0, \frac{1}{3}\}$	12	2	2
6	$\mathbf{C}^4$	$\{0, \frac{1}{3}\}$	40	2	3	17	$\mathbf{C}^4$	$\{0, \frac{1}{4}\}$	20	2	2
7	$\mathbf{C}^6$	$\{0, \frac{1}{4}\}$	126	2	3	18	$\mathbf{C}^5$	$\{0, \frac{1}{4}\}$	45	2	2
8	$\mathbf{C}^8$	$\{\frac{1}{9}\}$	64	1	2	19	$\mathbf{C}^9$	$\{0, \frac{1}{9}\}$	90	2	2
9	$\mathbf{H}^5$	$\{0, \frac{1}{4}\}$	165	2	3	20	$\mathbf{C}^{28}$	$\{0, \frac{1}{16}\}$	4060	2	2
10	$\mathbf{O}^3$	$\{0, \frac{1}{4}, \frac{1}{2}\}$	819	3	5	21	$\mathbf{H}^4$	$\{0, \frac{1}{4}\}$	36	2	2
11	$\mathbf{R}^{24}$	$\{0, \frac{1}{16}, \frac{1}{4}\}$	98 280	3	5	22	$\mathbf{H}^4$	$\{\frac{1}{9}, \frac{1}{3}\}$	64	2	2
						23	$\mathbf{R}^{16}$	$\{0, \frac{1}{9}\}$	256	2	2
						24	$\mathbf{C}^4$	$\{0, \frac{1}{4}, \frac{1}{2}\}$	60	3	3
						25	$\mathbf{R}^{16}$	$\{0, \frac{1}{16}, \frac{1}{4}\}$	2160	3	3
						26	$\mathbf{H}^3$	$\{0, \frac{1}{4}, \frac{1}{2}\}$	63	3	3
						27	$\mathbf{H}^3$	$\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3 \pm \sqrt{5}}{8}\right\}$	315	5	5
						28	$\mathbf{C}^{12}$	$\{0, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\}$	32 760	4	5

† For the values of n as given in the text. In case $\mathbf{C}^2$ we get $s = 1$, $t = 2$, with absolute bound attained.

Conclusions:

- applications: quantum cryptography, quantum state tomography, fidelity estimation
- open problems: existence, value of the bound

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Bibliography:
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