Families of generalized quantum measurements and defect of a unitary matrix

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2017 / 02 / 27

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preprint

joint work with

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to be published as "Quantum measurements with prescribed symmetry"

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table of contents

defect - derivation and properties

background

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2. $U = U^\dagger$

   additional constraint for U to have constant real diagonal

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U_1 and U_2 are **equivalent** if the following equality holds

$$U_1 = PDU_2D^\dagger P^{\mathrm{T}}$$

for some permutation P and diagonal unitary D matrix

background - dephasing

normalization $|U_{jk}| = 1$ for $j \neq k$

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$$U = \begin{bmatrix} c & U_{12} & ... & U_{1N} \\ * & c & ... & U_{2N} \\ ... & ... & ... & ... \\ * & * & ... & c \end{bmatrix} \longrightarrow$$

$$\mathrm{diag}[1, 1/U_{12}, ..., 1/U_{1N}] \begin{bmatrix} c & U_{12} & ... & U_{1N} \\ * & c & ... & U_{2N} \\ ... & ... & ... & ... \\ * & * & ... & c \end{bmatrix} \overline{\mathrm{diag}[1, 1/U_{12}, ..., 1/U_{1N}]} =$$

$$= \begin{bmatrix} c & \color{red}{1} & \color{red}{...} & \color{red}{1} \\ \color{red}{1} & c & ... & U_{2N} \\ \color{red}{...} & ... & ... & ... \\ \color{red}{1} & * & ... & c \end{bmatrix}$$

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absorbing $(N-1)$ phases: $U \longrightarrow DUD^\dagger$ ("**dephasing**")

problem

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given U (defined as above), is it possible to introduce a vector of phases $R = (R_1, R_2, ...) \in [0, 2\pi)^{\times}$ such that a family

$$U = U(R)$$

fulfills restricted conditions, or shall U be an isolated point? ...

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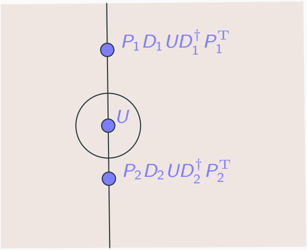
... as it is in the case of complex Hadamard matrices (CHM), for example:

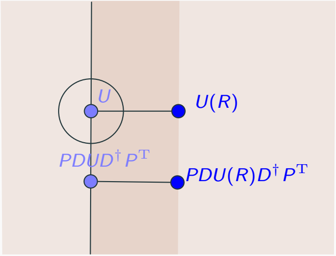
$$F_6 \longrightarrow F_6(R_1, R_2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega e^{iR_1} & \omega^2 e^{iR_2} & -1 & \omega^4 e^{iR_1} & \omega^5 e^{iR_2} \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & -e^{iR_1} & e^{iR_2} & -1 & e^{iR_1} & -e^{iR_2} \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 e^{iR_1} & \omega^4 e^{iR_2} & -1 & \omega^2 e^{iR_1} & \omega e^{iR_2} \end{bmatrix}$$

$\omega = e^{2\pi i/6}$

problem - isolation

isolation - in the neighbourhood of U there are no other INEQUIVALENT matrices





problem - visualisation

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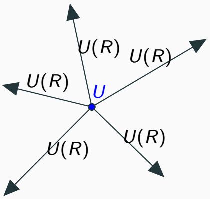

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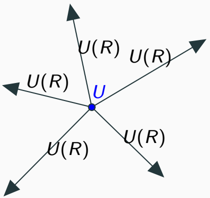

looking for "directions" $U(R)$ such that $U \longrightarrow U(R)$ with no 1^{st} order disturbances of the conditions:

1. $U(R)$ is unitary and hermitian
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3. $|U_{jk}(R)| = \text{constant}$ for $j \neq k$

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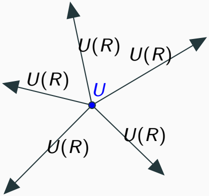

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consequences

- "directions" $U(R)$ belong to tangent space to $\mathbb{U}$ at U
- $U_{jk} \longrightarrow U_{jk} e^{iR_{jk}}$ for some anti-symmetric $R_{jk} = -R_{kj}$

space of directions - heuristic derivation

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consider a simple example $U \longrightarrow U(R) = U \circ e^{iR} \in \mathbb{C}^{3\times 3}$

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$U(R)U^\dagger(R) =$

$$= \begin{bmatrix} c & U_{12}e^{iR_{12}} & U_{13}e^{iR_{13}} \\ \overline{U_{12}}e^{-iR_{12}} & c & U_{23}e^{iR_{13}} \\ \overline{U_{13}}e^{-iR_{12}} & \overline{U_{23}}e^{-iR_{13}} & c \end{bmatrix}^2 =$$

$$= \begin{bmatrix} c^2 + |U_{12}|^2 + |U_{13}|^2 & * & * \\ 2cU_{12}e^{-iR_{12}} + U_{23}\overline{U_{13}}e^{i(R_{23}-R_{13})} & \text{constant} & * \\ \dots & \dots & \text{constant} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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general condition takes the form

$$2c\overline{U_{\square}}e^{-iR_{\square}} + \sum_{\square} U_{\square}\overline{U_{\square}}e^{i(R_{\square}-R_{\square})} = 0$$

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general condition takes the form, for $1 \leqslant j < k \leqslant N$

$$2c\overline{U_{jk}}e^{-iR_{jk}} + \sum_{l \neq j,k} U_{kl}\overline{U_{jl}}e^{i(R_{kl}-R_{jl})} = 0$$

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$g_U(R) = 0 \iff U(R) \in \mathbb{U}$

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linearization (differential) $Dg_U \implies$

$$-2cU_{kj}R_{jk} + \sum_{l\neq j,k} U_{kl}U_{lj}(R_{kl} - R_{jl}) = 0$$

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we are left with at most $\tau - r$ parameters (directions) where $\tau = \tau(N) = N(N-1)/2$

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$$\dim(\text{space of directions}) - \underbrace{(N-1)}_{\text{trivial phases}} - \underbrace{z(U)}_{\text{number of zeros above diagonal in } U}$$

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definition 2 - technical

defect$(U) \equiv$ dimension of the real solution space of the linear system

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with respect to the matrix variable R diminished by $(N-1) + z(U)$

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definition 3 - practical

$$\textbf{defect}(U) \equiv \underbrace{\tau(N)}_{\text{triangular number}} - \mathrm{rank}(R) - (N-1) - z(U)$$

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- Δ is invariant with respect to the **equivalence** relation:

$$\Delta(U) = \Delta(PDUD^{\dagger}P^{\mathrm{T}})$$

  for any permutation P and diagonal D matrix (preserving hermiticity!)

quantum measurement - ETF, SIC, MUB, ...

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- projective measurement (orthogonal projectors)
- positive-operator valued measure (POVM)

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$$\text{POVM} \supset \begin{cases} \text{MUB} \\ \text{SIC-POVM} \end{cases}$$

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- ETF (equiangular tight frame) of N vectors in $\mathbb{C}^d$

$$S^{\mathrm{ETF}} = \mathbb{I}_N + \frac{N-d}{d(N-1)} (\mathbb{O}_N - \mathbb{I}_N)$$

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- d^2 SIC-POVM (symmetric, informationally complete POVM) in $\mathbb{C}^d$

$$S^{\mathrm{SIC}} = S^{\mathrm{ETF}}\Big|_{N=d^2}$$

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- m MUB (mutually unbiased bases) in $\mathbb{C}^d$

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$\mathbb{O}_N$ - matrix of ones of size N

quantum measurement - elementary examples

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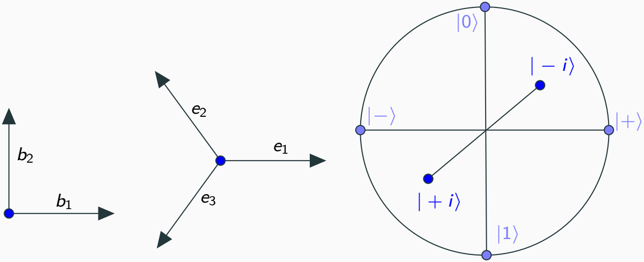

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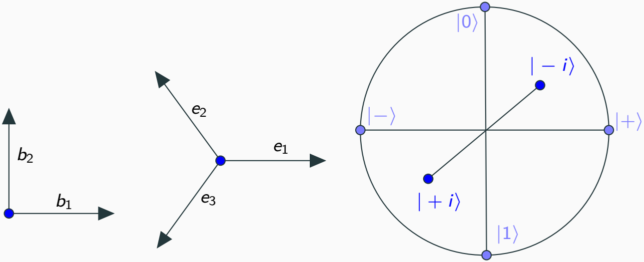

1. orthogonal basis: $\langle b_1|b_2\rangle = 0$

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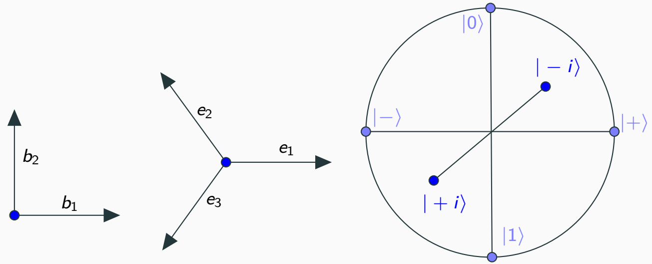

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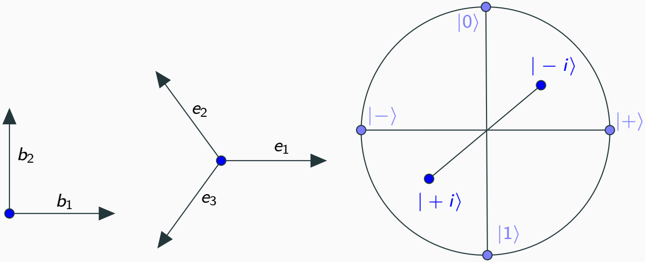

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2. maximal ETF on the plane: $e_k = e^{2\pi i (k-1)/3}$
3. simplest MUB:

   $\beta_1 = \{|0\rangle, |1\rangle\}$

   $\beta_2 = \{|+\rangle = |0\rangle + |1\rangle, |-\rangle = |0\rangle - |1\rangle\}$

   $\beta_3 = \{|+i\rangle = |0\rangle + i|1\rangle, |-i\rangle = |0\rangle - i|1\rangle\}$

   no normalization!

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quantum measurement - extension

problem

given generalised quantum measurement of predefined geometrical structure, is it possible to extend corresponding objects so that one gets a smooth family of POVM?

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1. motivation?
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2. one is surely supposed to use Δ but how?

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Gram matrix of a generic ETF

$$\mathcal{N} \begin{bmatrix} 1/\mathcal{N} & e^{i\alpha_{12}} & \dots & e^{i\alpha_{1N}} \\ * & 1/\mathcal{N} & \dots & e^{i\alpha_{2N}} \\ \dots & \dots & \dots & \dots \\ * & * & \dots & 1/\mathcal{N} \end{bmatrix}$$

$\mathcal{N} = d(N-1)/(N-d)$

Gram matrix of a set of m MUB $\{\mathbb{I}_d, H_1, \dots, H_m\}$ each of size d

$$\begin{bmatrix} \mathbb{I}_d & H_1 & H_2 & \dots & H_m \\ H_1^\dagger & \mathbb{I}_d & H_1^\dagger H_2 & \dots & H_1^\dagger H_m \\ H_2^\dagger & H_2^\dagger H_1 & \mathbb{I}_d & \dots & H_2^\dagger H_m \\ \dots & \dots & \dots & \dots & \\ H_m^\dagger & H_m^\dagger H_1 & H_m^\dagger H_2 & \dots & \mathbb{I}_d \end{bmatrix}$$

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observation

POVM with predefined symmetry $\Longleftrightarrow$ unitary and hermitian matrix with constant diagonal

results - defect and POVM with predefined symmetry

Δ(MUB)

result 1 of 4

maximal sets of $d+1$ MUB are <u>isolated</u> in dimension $d \in \{4, 8, 9, 16\}$

$m \setminus d$	2	3	4	5	6	7	8	9	...	16
2	0	0	Δ_4	0	Δ_6	0	Δ_8	Δ_9	...	Δ_{16}
3	0	0	$\times_{4,3}$	0	$\times_{6,3}$	0	$\times_{8,3}$	$\times_{9,3}$	...	$\times_{16,3}$
4		0	0	0	?	0	$\times_{8,4}$	$\times_{9,4}$	...	$\times_{16,4}$
5			0	0	?	0	$\times_{8,5}$	$\times_{9,5}$	...	$\times_{16,5}$
6				0	?	0	$\times_{8,6}$	$\times_{9,6}$	...	$\times_{16,6}$
7					?	0	$\times_{8,7}$	$\times_{9,7}$	...	$\times_{16,7}$
8						0	0	$\times_{9,8}$	...	$\times_{16,8}$
9							0	0	...	$\times_{16,9}$
10								0	...	...
...									...	...
16									...	0
17										0

Δ(MUB)

result 1 - comments

1. $d \in \mathbb{P}^k \longrightarrow$ well known cases...
2. Δ_d values pertain only to a particular MUB construction

observation

if $m = 2$ then Δ coincides with the standard defect for CHM, because

$$G = \begin{bmatrix} \mathbb{I}_d & H \\ H^\dagger & \mathbb{I}_d \end{bmatrix}$$

$\Delta_8 = 21 = d(F_2 \otimes F_2 \otimes F_2)$
$\Delta_8 = 13 = d(F_4 \otimes F_2)$
and so on...

3. $\times_{d,m} \geqslant 0$ values pertain... too, however for given construction they remain constant regardless of the subset of MUB
4. we used [2] and [3] to construct the sets of MUB

Δ(SIC)

result 2 of 4

SIC-POVM are isolated in dimensions $d \in \{4, ..., 16\}$

result 2 - comments

1. SIC were obtained by means of fiducial vectors [4]
2. Hoggar lines form an isolated structure
3. $\Delta(\mathrm{SIC}) \neq 0$ for $d = 3$ which is in accordance with the existence of well known $1-$parameter family

$$\phi(\alpha) = \frac{1}{2}[1, e^{i\alpha}, 0]^{\mathrm{T}} : \alpha \in [0, 2\pi)$$

$\alpha = \pi \implies \Delta = 4$

$\alpha \neq \pi \implies \Delta = 2$

Δ(ETF) - Fourier case

result 3 of 4

Δ(ETF) of $N = n^2$ vectors in dimension $d = n(n-1)/2$ - hermitian counterpart of classic Fouriers

let $[F_N]_{jk} = \omega^{\mathrm{mod}(j,n)\lfloor \frac{k}{n} \rfloor - \mathrm{mod}(k,n)\lfloor \frac{j}{n} \rfloor}$

for $j, k = 0, 1, ..., \underline{N - 1 = n^2 - 1}$ where $\omega = e^{2\pi i/n}$

---

by construction:

- $F = F^\dagger = F^{-1}$
- $F_N = (F_n \otimes F_n)P$ for some permutation matrix P

and

$$p \in \mathbb{P} \Longrightarrow \Delta(F_{p^2}) = \frac{1}{2}(p-2)(p-1)(p+1)$$
$$n \in \mathbb{N} \setminus \mathbb{P} \Longrightarrow \Delta(F_{n^2}) > 0$$

Δ(ETF)

result 4 of 4

Δ returns reasonable results for many unitary and hermitian matrices

example - 1-parametric family

$$C_6(b) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & e^{ib} & 1 & -e^{ib} \\ 1 & -1 & 0 & -e^{ib} & 1 & e^{ib} \\ 1 & e^{-ib} & -e^{-ib} & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & -e^{-ib} & e^{-ib} & 1 & -1 & 0 \end{bmatrix}$$

$\Delta(C_6(0)) = 4$ and

$\Delta(C_6(b)) = 1$ for generic value of $b \in (0, 2\pi)$

matrix taken from [5]

summary

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- extended research tool for seeking smooth families of POVM with predefined symmetry
- full sets of MUB are isolated in dimension: 4, 8, 9 and 16
- SIC-POVM are isolated from dimension 4 to 16
- Hoggar lines (3-qubit system) are isolated
- upper bound for the maximal number of parameters for several MUB and ETF

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Thank You