# Pure states that are 'most quantum' with respect to a given generalized measurement

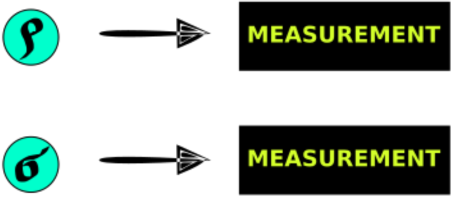
Anna Szymusiak

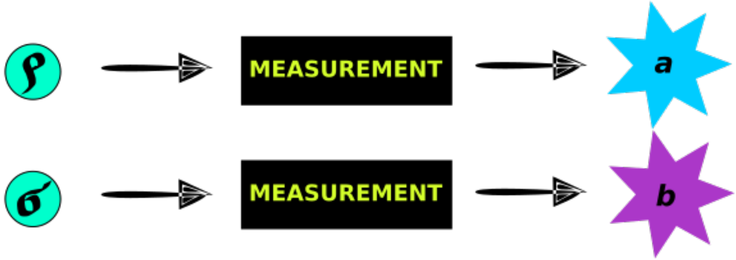
Institute of Mathematics, Jagiellonian University, Kraków

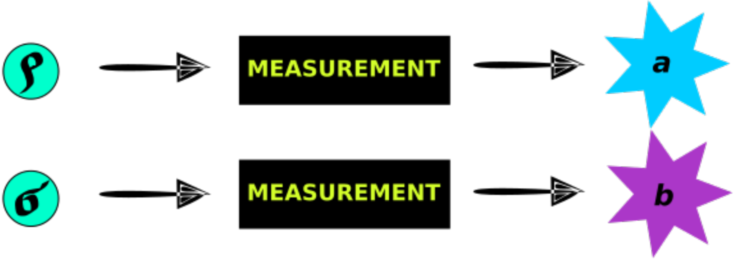
24.10.2016

# Outline

# In what sense one quantum state can be ‘more’ or ‘less quantum’ than the other one?


MEASUREMENT
MEASUREMENT


ρ
MEASUREMENT
a
σ
MEASUREMENT
b




measure of **'quantumness'**

=

measure of **randomness** of the measurement outcomes

## measuring randomness – the Shannon entropy

### Definition

The **Shannon entropy** of probability distribution $P$ on finite space $X := \{x_1, \ldots, x_k\}$ is given by:

$$H(P) := \sum_{j=1}^{k} \eta(P(x_j)),$$

where $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

- $H$ is concave on the probability simplex
- $H(P) \leq \ln k$ and equality holds iff $P$ is uniform
- $H(P) \geq 0$ and equality holds iff $P$ is the Dirac delta

## Definition

A discrete quantum measurement is described by a **positive operator valued measure** (POVM) $\Pi = \{\Pi_j\}_{j=1}^{k}$, where $\Pi_j$ are positive semidefinite operators on $\mathbb{C}^d$ that satisfy the identity decomposition

$$\sum_{j=1}^{k} \Pi_j = \mathbb{I}.$$

If the state before measurement was $\rho$, the probability of obtaining $j$-th outcome is given by $\mathrm{tr}(\rho\Pi_j)$.

## Definition

The **Shannon entropy of measurement** $\Pi$ is defined as the Shannon entropy of the probability distribution of the measurement outcomes:

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho\Pi_j)),$$

for an initial state $\rho$, where $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho \Pi_j)),$$

- $H(\cdot, \Pi)$ attains its minima in pure states.
- If $\mathrm{tr}(\Pi_j) = \frac{d}{k}$ for $j = 1, \ldots, k$, then $H(\rho, \Pi)$ is maximal for maximally mixed state $\rho_* := \frac{1}{d}\mathbb{I}$.
- Taking maximum over pure states only – how 'badly' can we end by choosing initially any pure state.
- The minimizers of $H(\cdot, \Pi)$ can be interpreted as 'the most classical' (with respect to $\Pi$) and the maximizers as 'the most quantum' (wrt $\Pi$).
- Instead of $H(\rho, \Pi)$ one can consider $\widetilde{H}(\rho, \Pi) = \ln k - H(\rho, \Pi)$, the relative entropy of $\Pi$ (with respect to a uniform distribution).

Similar problems, concerning maximal ‘quantumness’, studied already:

- **‘Queens of Quantum’**: maximizers of the Hilbert-Schmidt distance from the convex hull of coherent states

  Giraud, O., Braun, P., Braun, D., New J. Phys. **12**, 063005 (2010)

- maximizers of the Wehrl entropy

  Bæcklund, A., Bengtsson, I., arXiv:1312.2427 [quant-ph]

*The solutions for both problems coincide for dimensions 2-8 and 10, but not for 9.*

## Fact

If $\Pi$ is a projective valued measure (PVM), then $\min_{\rho \in \mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho, \Pi) = 0$.

## Proof.

$\Pi := \{|e_j\rangle\langle e_j|\}_{j=1}^d$, where $\{|e_j\rangle\}_{j=1}^d$ is an orthonormal basis in $\mathbb{C}^d$. Take $(1/\sqrt{d}) \sum_{j=1}^d e^{i\theta_j} |e_j\rangle$, where $\theta_j \in \mathbb{R}$. □

## Definition

POVM $\Pi$ is called informationally complete (IC-POVM) if the conditions $\mathrm{tr}(\rho\Pi_j) = \mathrm{tr}(\sigma\Pi_j)$ for $j = 1, \ldots, k$ imply $\rho = \sigma$ for every states $\rho, \sigma \in \mathcal{S}(\mathbb{C}^d)$.

## Fact

If the rank-1 normalized $\Pi = \{\Pi_j\}_{j=1}^{k}$ is informationally complete, then

$$\min_{\rho\in\mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho, \Pi) > 0. \quad (1)$$

Moreover, if $d = 2$, the converse is also true.

## Proof.

If $\widetilde{H}(\rho, \Pi) = 0$, the probability distribution of the measurement outcomes is uniform. By the IC of $\Pi$: $\rho = \rho_*$ and so $\min_{\rho\in\mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho, \Pi) > 0$.

$d = 2$: If $\Pi$ is not IC, the corresponding Bloch vectors span 2-dim subspace. Take the orthogonal Bloch vector. □

## Theorem

*Let $\Pi$ be an informationally complete highly symmetric POVM (HS-POVM) in dimension two, but not a SIC-POVM. Then the entropy (resp. relative entropy) of $\Pi$ restricted to the set of pure states attains its maximum (resp. minimum) value exactly in the states which Bloch vectors correspond to*

1) *the vertices of the dual polyhedron, if $\Pi$ is represented by a platonic solid,*
2) *the vertices of the octahedron, if $\Pi$ is represented by cuboctahedron,*
3) *the vertices of the icosahedron, if $\Pi$ is represented by icosidodecahedron.*

## Sketch of the proof.

- Hermite interpolation method
- invariant polynomials □

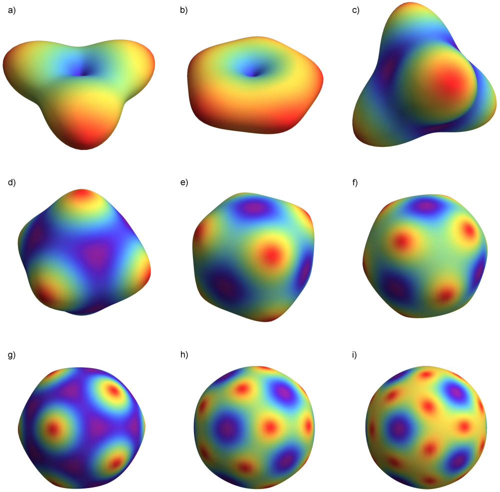



Figure: The relative entropy of highly symmetric qubit measurements. The rainbow-colors scale that ranges from *red* (maximum) to *purple* (minimum) is used.

## Definition

A **symmetric informationally complete POVM (SIC-POVM)** consists of $d^2$ subnormalized rank-one projections $\Pi_j = |\phi_j\rangle\langle\phi_j|/d$ with equal pairwise Hilbert-Schmidt inner products:

$$\mathrm{tr}(\Pi_i^* \Pi_j) = \frac{|\langle\phi_i|\phi_j\rangle|^2}{d^2} = \frac{1}{d^2(d+1)} \text{ for } i \neq j$$

.

- first studied in the context of *equiangular lines* in $\mathbb{C}^d$ or *complex spherical 2-designs* (e.g. Hoggar's papers 1978-82)
- extensively examined by Zauner in his PhD Thesis (1999), independently studied by Renes et al. (2003)
- the existence of SIC-POVMs in every dimension – still an open problem
  - analytical solutions known for $d = 2 - 16, 19, 24, 28, 35, 48$
  - numerical confirmation up to $d \approx 300$
- $d = 3$ – infinite family of nonequivalent SIC-POVMs
- all known SIC-POVMs are group-covariant
  - the Hoggar SIC-POVM ($d = 8$) is the only known SIC-POVM that is not group-covariant with respect to the finite Weyl-Heisenberg group
  - prime dimensions: group-covariance implies WH-covariance (Zhu, 2010)
  - there are exactly 3 *supersymmetric* SIC-POVMs (any two rays can be transformed into any two rays via symmetry group's action): $d = 2$, $d = 3$

## Theorem

*Let* $\Pi = \{(1/d)|\phi_j\rangle\langle\phi_j|\}_{j=1}^{d^2}$ *be a SIC-POVM in dimension* $d$*. Then states* $|\phi_j\rangle\langle\phi_j|$ *for* $j = 1, \ldots, d^2$ *are the only minimizers of the relative entropy restricted to the pure states and*

$$\min_{\rho\in\mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho, \Pi) = \widetilde{H}(|\phi_j\rangle\langle\phi_j|, \Pi) = \ln d - \frac{d-1}{d}\ln(d+1), \tag{2}$$

*for* $j = 1, \ldots, d^2$*. Moreover,* $\min_{\rho\in\mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho, \Pi) \xrightarrow{d\to\infty} 0$*.*

## Sketch of the proof.

- The Hermite interpolation in the points $1/d$ and $1/(d(d+1))$ (up to 0-th and 1-st derivative respectively) gives us a polynomial approximating $\eta$ **from above** of degree at most 2.
- Alternatively: the obtained probability distribution is a maximizer of the Shannon entropy over all probability distributions $(p_1, \ldots, p_{d^2})$ such that $\sum_{i=1}^{d^2} p_i^2 = 2/(d(d+1))$.

Harremoës, P., Topsøe, F., IEEE Trans. Inform. Theory **47**, 2944–2960 (2001) □

While we know the minimum and maximum values of the relative entropy of some POVMs, it would be worth taking a look at its average:

$$\left\langle \widetilde{H}(\rho, \Pi) \right\rangle_{\rho \in \mathcal{P}(\mathbb{C}^d)} = \int_{\mathcal{P}(\mathbb{C}^d)} \left( \ln d - \frac{d}{k} \sum_{j=1}^{k} \eta \left( \operatorname{Tr} \left( \rho \rho_j \right) \right) \right) \mathrm{d}m_{FS}\left(\rho\right) \tag{3}$$

$$= \ln d - d \left( \int_{\mathcal{P}(\mathbb{C}^d)} \eta \left( \operatorname{Tr} \left( \rho \rho_1 \right) \right) \mathrm{d}m_{FS}\left(\rho\right) \right)$$

$$= \ln d - \sum_{j=2}^{d} \frac{1}{j} \to 1 - \gamma \quad (d \to \infty),$$

where $\gamma \approx 0.57722$ is the Euler-Mascheroni constant.

Surprisingly, the average value of relative entropy over all pure states depends only on the dimension $d$.

Jones, K.R.W., J. Phys. A **24**, 121–130 (1991)

| convex hull of the orbit | informational power (max relative entropy) | min relative entropy |
|---|---|---|
| digon | 0.69315 | 0 |
| regular $n$-gon ($n \to \infty$) | 0.30685 | 0 |
| tetrahedron | 0.28768 | 0.14384 |
| octahedron | 0.23105 | 0.17744 |
| cube | 0.21576 | 0.17744 |
| cuboctahedron | 0.20273 | 0.18443 |
| icosahedron | 0.20189 | 0.18997 |
| dodecahedron | 0.19686 | 0.18997 |
| icosidodecahedron | 0.19486 | 0.19099 |
| average relative entropy | 0.19315 | |

Table: The approximate values of informational power (maximum relative entropy) and minimum relative entropy on pure states (up to five digits) for all types of HS-POVMs in dimension two.

| dimension | informational power (upper bound) | average value of relative entropy | min relative entropy |
|---|---|---|---|
| 2 | 0.28768 | 0.19315 | 0.14384 |
| 3 | 0.40547 | 0.26528 | 0.17442 |
| 4 | 0.47000 | 0.30296 | 0.17922 |
| 5 | 0.51083 | 0.32611 | 0.17603 |
| 6 | 0.53900 | 0.34176 | 0.17017 |
| $d$ | $\ln \frac{2d}{d+1}$ | $\ln d - \sum_{j=2}^{d} \frac{1}{j}$ | $\ln d - \frac{d-1}{d} \ln(d+1)$ |
| $d \to \infty$ | $\ln 2 \approx 0.69315$ | $1 - \gamma \approx 0.42278$ | 0 |

Table: The approximate values of informational power (upper bounds for $d > 3$), average and minimum relative entropy on pure states (up to five digits) for SIC-POVMs.

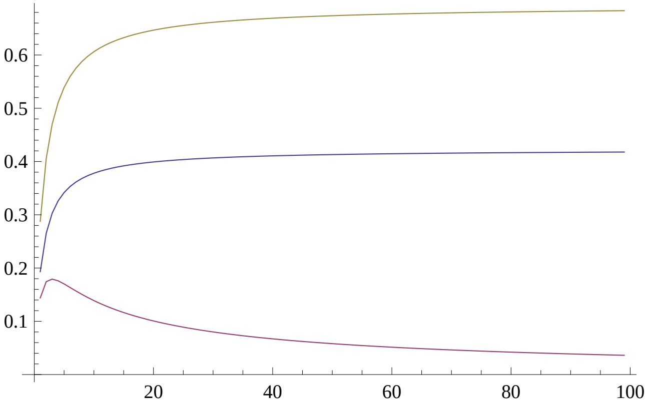



Figure: The upper bound for informational power (*dark yellow*), average value of relative entropy (*violet*) and minimum relative entropy on pure states (*purple*) of SIC-POVMs in dimensions from 2 to 100.

# Connections with the entropic **certainty** relations

Combine $m$ normalized rank-1 POVMs $\Pi^i = (\Pi^i_j)_{j=1,\ldots,k}$ $(i = 1, \ldots, m)$ to obtain another normalized rank-1 POVM $\Pi := (\frac{1}{m}\Pi^i_j)^{i=1,\ldots,m}_{j=1,\ldots,k}$.

Then the entropic certainty relation

$$\frac{1}{m}\sum_{i=1}^{m} H(\rho, \Pi^i) \leq \mathcal{B}_{max}$$

can be expressed equivalently as

$$H(\rho, \Pi) \leq \mathcal{B}_{max} + \ln m$$

Let $\Pi^i$ be a PVM, i.e. $\Pi^i_j := |e^i_j\rangle\langle e^i_j|$ for some orthonormal basis $\{e^i_j\}_{j=1}^d$ such that $\Pi = (\frac{1}{d+1}\Pi^i_j)_{j=1,\ldots,k}^{i=1,\ldots,m}$ is a complete MUB measurement.

$$\frac{1}{d+1}\sum_{i=1}^{d+1} H(\rho,\Pi^i) \leq \mathcal{B}_{max} \leq \mathcal{B}_{max}^{SR} := \ln d - \frac{(d-1)^2\ln(d-1)}{(d-2)d(d+1)}$$

Sánchez-Ruiz, J., Phys. Lett. A **201**, 125–131 (1995)

Note that for $\rho_0 := |e^i_j\rangle\langle e^i_j|$ we get

$$\frac{1}{d+1}\sum_{i=1}^{d+1} H(\rho_0,\Pi^i) = d(\ln d + \frac{d}{d+1}\ln(d+1))$$

which gives us

$$\mathcal{B}_{max} \geq d(\ln d + \frac{d}{d+1}\ln(d+1)).$$

Moreover, we know that

$$\left\langle \frac{1}{d+1}\sum_{i=1}^{d+1} H(\rho,\Pi^i)\right\rangle_{\rho\in\mathcal{P}(\mathbb{C}^d)} = \Psi(d+1)-\Psi(2) \approx \ln d - (1-\gamma)$$

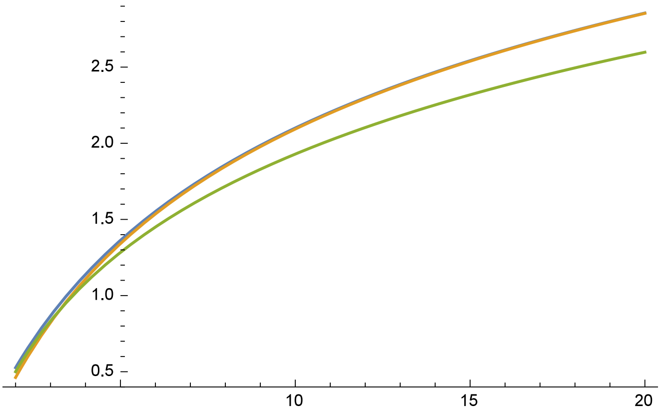



Figure: The Sanchez-Ruiz upper bound (*blue*), the lower bound for $\mathcal{B}_{max}$ (*orange*) and the average entropy (*green*) for complete MUB (if they exist).

Is it true that (for $d > 3$) the 'most quantum' pure states for MUB are defined by the MUB itself?

| POVM | MIN | MAX |
|---|---|---|
| 2-SIC | twin 2-SIC | 2-SIC |
| Hesse 3-SIC | complete 3-MUB | Hesse 3-SIC |
| generic 3-SIC | 3-ONB | generic 3-SIC |
| Hoggar 8-SIC | twin Hoggar 8-SIC | Hoggar 8-SIC |
| complete 2-MUB | complete 2-MUB | two twin 2-SICs |
| complete 3-MUB | Hesse 3-SIC | ? |
| cube (two twin 2-SICs) | cube | complete 2-MUB |
| icosahedron | icosahedron | dodecahedron |
| dodecahedron | dodecahedron | icosahedron |
| cuboctahedron | cuboctahedron | complete 2-MUB |
| icosidodecahedron | icosidodecahedron | icosahedron |
| d-SIC | ? | d-SIC |
| complete d-MUB (d>3) | ? | *complete d-MUB ???* |
| regular n-gon (n>2) | dual n-gon | digon (2-ONB) |
| d-ONB | d-ONB | (d-1)-torus |

Table: Extremal configurations for some POVMs

# Conclusions

- Maximizers of entropy over pure states for HS-POVMs in dimension 2
- Maximizers for SIC-POVMs in any dimension
- Connection with entropic certainty relations
- Duality in the extremal configurations – under which conditions?
- Is it possible to say something about the variance of entropy? Does it tend to 0 (e.g. for MUB)?

# THANK YOU

## The optimization method based on the Hermite interpolation

- a sequence of points: $a \leq t_1 < t_2 < \ldots < t_m \leq b$
- a sequence of positive integers: $k_1, k_2, \ldots, k_m$
- a real valued function $f \in \mathcal{C}^D([a,b])$, where $D := k_1 + k_2 + \ldots + k_m$

There exists a unique polynomial $p$ of degree less than $D$ that agree with $f$ at $t_i$ up to a derivative of order $k_i - 1$ (for $1 \leq i \leq m$), that is,

$$p^{(k)}(t_i) = f^{(k)}(t_i), \qquad 0 \leq k < k_i. \tag{4}$$

We will need the following formula for the error in Hermite interpolation:

**Lemma**

*For each $t \in (a,b)$ there exists $\xi \in (a,b)$ such that $\min\{t, t_1\} < \xi < \max\{t, t_m\}$ and*

$$f(t) - p(t) = \frac{f^{(D)}(\xi)}{D!} \prod_{i=1}^{m} (t - t_i)^{k_i}. \tag{5}$$

Assume that all the derivatives of $f$ of even order are strictly negative in $(a, b)$ and these of odd order greater than 1 are strictly positive (holds true for $\eta$):

$$f^{2l}(x) < 0, \quad f^{2l+1}(x) > 0 \quad \text{for } x \in (a, b),\ l = 1, 2, \ldots . \tag{6}$$

Moreover, let us assume that

$$k_i := \begin{cases} 1, & \text{if } t_i \in \{a, b\} \\ 2, & \text{otherwise} \end{cases} . \tag{7}$$

## Observation

*Under above assumptions, the Hermite polynomial p interpolates f*

1. *from below, if $t_1 = a$,*
2. *from above, if $t_1 > a$.*

*Moreover, $f(t) = p(t)$ if and only if $t = t_i$ for some $i = 1, \ldots, m$.*

## Proof.

1. Consider two cases:

- $t_m = b$
  We have $D = 2m - 2$, so $f^{(D)}(\xi) < 0$ for $\xi \in (a, b)$. We also get

$$\prod_{i=1}^{m}(t - t_i)^{k_i} = (t - a)(t - b)\prod_{i=2}^{m-1}(t - t_i)^2 \leq 0, \quad (8)$$

  for $t \in [a, b]$.

- $t_m < b$
  We have $D = 2m - 1$ and so $f^{(D)}(\xi) > 0$ for $\xi \in (a, b)$. Moreover,

$$\prod_{i=1}^{m}(t - t_i)^{k_i} = (t - a)\prod_{i=2}^{m}(t - t_i)^2 \geq 0 \quad (9)$$

  for $t \in [a, b]$.

Finally, we apply (5) to get $f(t) \geq p(t)$ for $t \in [a, b]$ with equality exactly in the points $t_i$ for $i = 1, \ldots, m$.

2. The proof is analogous. □

How can we apply this method in our situation?

- $B = \{v_j | j = 1, \ldots, k\} \subset B(d)$ – the set of Bloch vectors corresponding to a rank-1 normalized POVM
- Define $F : B(d) \to \mathbb{R}$ by $F(u) := \sum_{j=1}^{k} f(u \cdot v_j)$, where $f : [-1/(d-1), 1] \to \mathbb{R}$ fulfills the condition given in (6). (In case of the Shannon entropy $f(t) = \eta(((d-1)t + 1)/d)$ .)
- For the set of the interpolation points take $T := \{w \cdot v_j | j = 1, \ldots, k\}$, where $w \in B(d)$.

Then, by Observation 1, if $-1/(d-1) \in T$, we get for every $u \in B(d)$

$$F(u) = \hat{g}\left(\sum_{j=1}^{k} f(u \cdot v_j)\right) \geq \hat{g}\left(\sum_{j=1}^{k} p(u \cdot v_j)\right) =: P(u) \qquad (10)$$

with equality for $u = w$ (and any other $u$ satisfying $\{u \cdot v_j | j = 1, \ldots, k\} \subset T$). Similarly, if $-1/(d-1) \notin T$ we get $F(u) \leq P(u)$ with equality for $u$ as above. In consequence, if $-1/(d-1) \in T$ (respectively $-1/(d-1) \notin T$) and $P$ attains its global minimum (maximum) in $w$, then $w$ is also a global minimizer (maximizer) of $F$, since from (10) we get $F(u) \geq P(u) \geq P(w) = F(w)$ for every $u \in B(d)$.