# ON Maximal Entanglement

CODES, HOLOGRAPHY, QMA

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Quantum Information is bringing new insights

Traditional emphasis on operators

$$H|\psi\rangle = E|\psi\rangle$$

QI emphasis on states

Example: Quantum Phase Transitions

$$H$$

Criticality
RG flows on coupling constants
OPE,Conformal Symmetry

$$|\psi\rangle$$

Scaling of entropy
RG flows on states
Distribution of entanglement: MERA

Von Neumann entropy $\rho_A = Tr_B |\psi\rangle_{AB}\langle\psi|$ $S(\rho_A) = Tr\,\rho_A \log \rho_A$

*Fixed points: scaling with the size <u>L</u>*

$$S(\rho_A) = \frac{c}{3} \log L \ll L$$

Callan-Wilczek 94, Vidal-jil-Rico-Kitaev 02

| | |
|---|---|
| $S \sim n - const$ | Random states, QMA problems,<br>Local translational invariant higher d |
| $S \sim .8858\, n + const$ | Prime state |
| $S \sim n^{\frac{d-1}{d}} + const$ | Area law in d-dimensions |
| $S \sim \frac{c}{3} \log n + const$ | Critical scaling in d=1 |
| $S \sim \log(\xi) = const$ | Finitely correlated states<br>away from criticality |

# Outline

Introduction

Absolute Maximal Entanglement

H-code

The most entangled spin chain

Gauge theory from MaxEnt

# Absolute Maximal Entanglement

Def: Absolute Maximal Entanglement

$$|\psi\rangle \in H_d^{\otimes n}$$

$$\rho_m = Tr_{n-m} |\psi\rangle\langle\psi| = \frac{1}{d^m} I_{d^m} \qquad \forall m$$

# Absolute Maximum Entanglement

Helwig, Cui, JIL, Riera, Lo (2012)

Multipartite teleportation
Quantum secret sharing
Reed-Solomon codes

Helwig thesis (2014)

## Holography

JIL, Sierra (2015); Pastawski, Yoshida, Harlow, Preskill (2015)

## Combinatorial design

Goyneche, Alsina, JIL, Riera, Zyczkowsky (2015)

Previous work on perfect states, balanced states, LU, LOCC,...

# 4-qubit maximally entangled state?

NO: **there is no 4-qubit state maximally entangled in all its partitions**

Monogamy of entanglement vs local dimensionality

Scott, Facchi, Parisi, …

AME exist for n=2,3,5,6 qubits
AME do not exist for n=4 and n>7
AME for n=7 is not known, though numerical evidence is against it

# 4-qutrit Absolute Maximally Entangled state in all partitions

- First non-trivial generalization of EPR states
- Symmetric (mod relabeling)
- Higher local dimensionality leaves space for entanglement
- Maximal surprise = Error correction

$$|P\rangle_{ABCS} = \frac{1}{3}\sum_{i,j=0,1,2} |i, j, i+j, i+2j\rangle$$

$$\rho_{AB} = \rho_{AC} = \rho_{AS} = \frac{1}{9} I_9$$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ mod 3 notation

$$|AME(4,3)\rangle_{ABCS} = \sum_{i,\, j=0,1,2} |i\,, j\,, i+j\,, i+2j\rangle$$

| $i$ | $j$ | $i+j$ | $i+2j$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 2 |
| 0 | 2 | 2 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 2 | 0 |
| 1 | 2 | 0 | 2 |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 2 | 2 | 1 | 0 |

**basis** → **basis**

same for any partition

$$t_{abcs} = A^{s}_{a\,bc}$$

*s,a* fixed by *b,c*

Hamming distance = 3

(error correction code, Reed-Solomon 1950)

# AME and Reed-Solomon codes

$$c_i = e_i \cdot G$$

codewords      basis

$$AME(4,3) \qquad G = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$AME(n, d)$$

$$q = d = Prime$$

$$n = d + 1 \qquad k = n/2$$

$$G = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 \\ a_1 & a_2 & \ldots & a_q & 0 \\ . & . & \ldots & . & 0 \\ a_1^k & a_2^k & \ldots & a_q^k & 1 \end{pmatrix}$$

Multi-Unitarity

| $i$ | $j$ | $i+j$ | $i+2j$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 2 |
| 0 | 2 | 2 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 2 | 0 |
| 1 | 2 | 0 | 2 |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 2 | 2 | 1 | 0 |

$$t_{abcd} = U_{(ab)(cd)} = U_{(ac)(bd)} = U_{(ad)(bc)}$$

$$|AME\rangle = \sum_{i_1,\dots,i_m} |i_1,\dots,i_m\rangle\, U_{(i_1,\dots,i_m),(i_{m+1},\dots,i_n)} |i_{m+1},\dots,i_n\rangle$$

$$U\,U^{+} \sim I\ , isometry$$

# AME support

Local Unitary (LU) invariance allows for a minimal support

$AME(4,3)$ needs only 9 superposed elements

$AME(6,2)$ needs a minimum of 16 superposed elements

# AME

Are AME related to any maximal violation of Bell Inequalities?

Are AME related to some underlying entanglement structure of physical systems?

Is Maximal Entropy possible in Condensed Matter systems?

# Holographic codes

# Tensor Networks

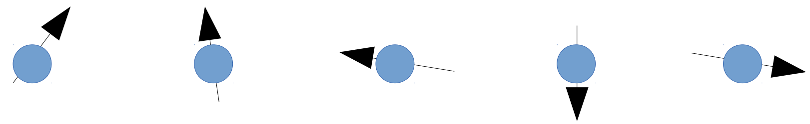
Simulation of quantum systems
on a classical computer

**Exact Simulation** | **Monte Carlo** | **Tensor Networks**

Tensor Netwoks

- Variational approach
- Escape Sign problem
- Support area law entanglement
- Time-evolution
- Libraries: DMRG, iTEBD,..
- Include symmetries

Example: Matrix Product State

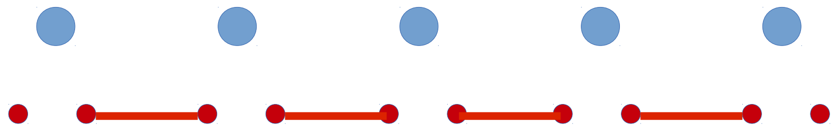


$$|\psi\rangle = \sum_{i_1 \dots i_n} c^{i_1 \dots i_n} |i_1 \dots i_n\rangle \in (C^2)^{\otimes n}$$

$2^n$ *coefficients*

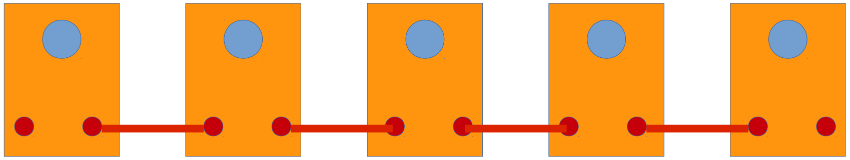
$$c^{i_1 \dots i_n} = Tr\, A^{i_1} \dots A^{i_n}$$

Populate the system with entangled pairs



$$|\varphi\rangle = \sum_{\alpha = 1, \dots, \chi} |\alpha\, \alpha\rangle$$

and project ancillary states onto physical indexes



Just capture correlations!!

$A^i_{\alpha\beta}$

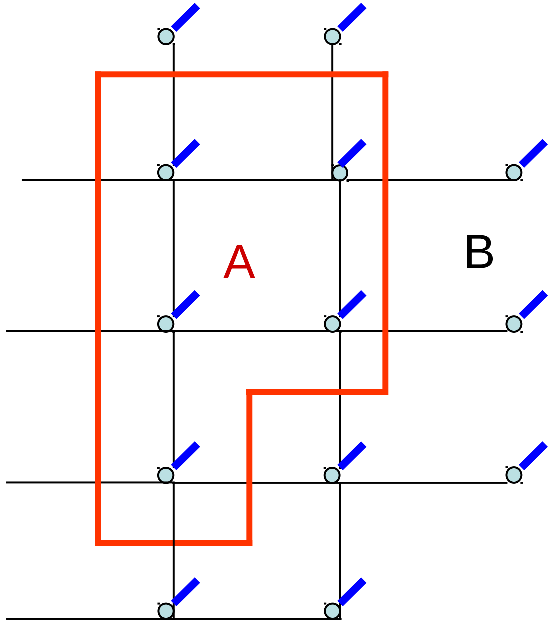
$$c^{i_1 \dots i_n} = Tr\, A^{i_1} \dots A^{i_n}$$

Exponential Kolmogorov complexity reduction!

$d^n\ coefficients$ → $n\, d\, \chi^2\ coefficients$

Adaptive algorithm to entanglement

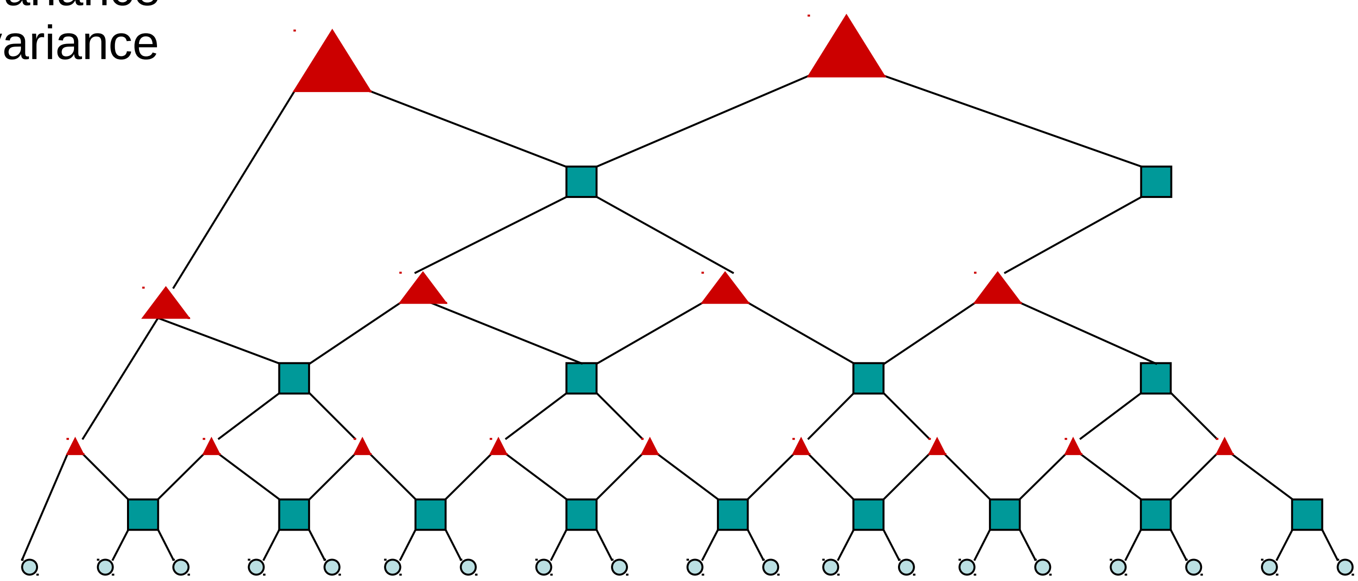
| | | |
|---|---|---|
| $\chi = 1$ | Product state | |
| $\chi = poly(n)$ | Slight entanglement | 1D quantum phase transitions |
| $\chi = 2^{\frac{n}{2}}$ | Maximal entanglement | Problems in QMA |




Support for entanglement entropy
is proportional to the boundary

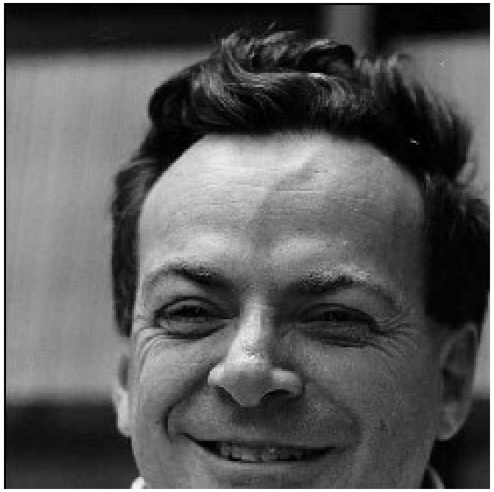
$$S_A = S_B \propto L$$ Area law

| | | |
|---|---|---|
| MPS | D=1 | translational invariance |
| PEPS | D=2,3 | translational invariance |
| Tree | | scale invariance |
| MERA | | scale invariance |



Cirac, Verstraete
Vidal

Tensor Networks are compressors of correlations



$$|\psi\rangle = \sum_{i_1 \dots i_4 = 1,\dots,9} c^{i_1 \dots i_4} |i_1 \dots i_4\rangle \in (C^9)^{\otimes 4}$$

level of grey     pixel address

$$c^{i_1 \dots i_n} = Tr\, A^{i_1} \dots A^{i_n} \qquad dim\, A = \chi$$

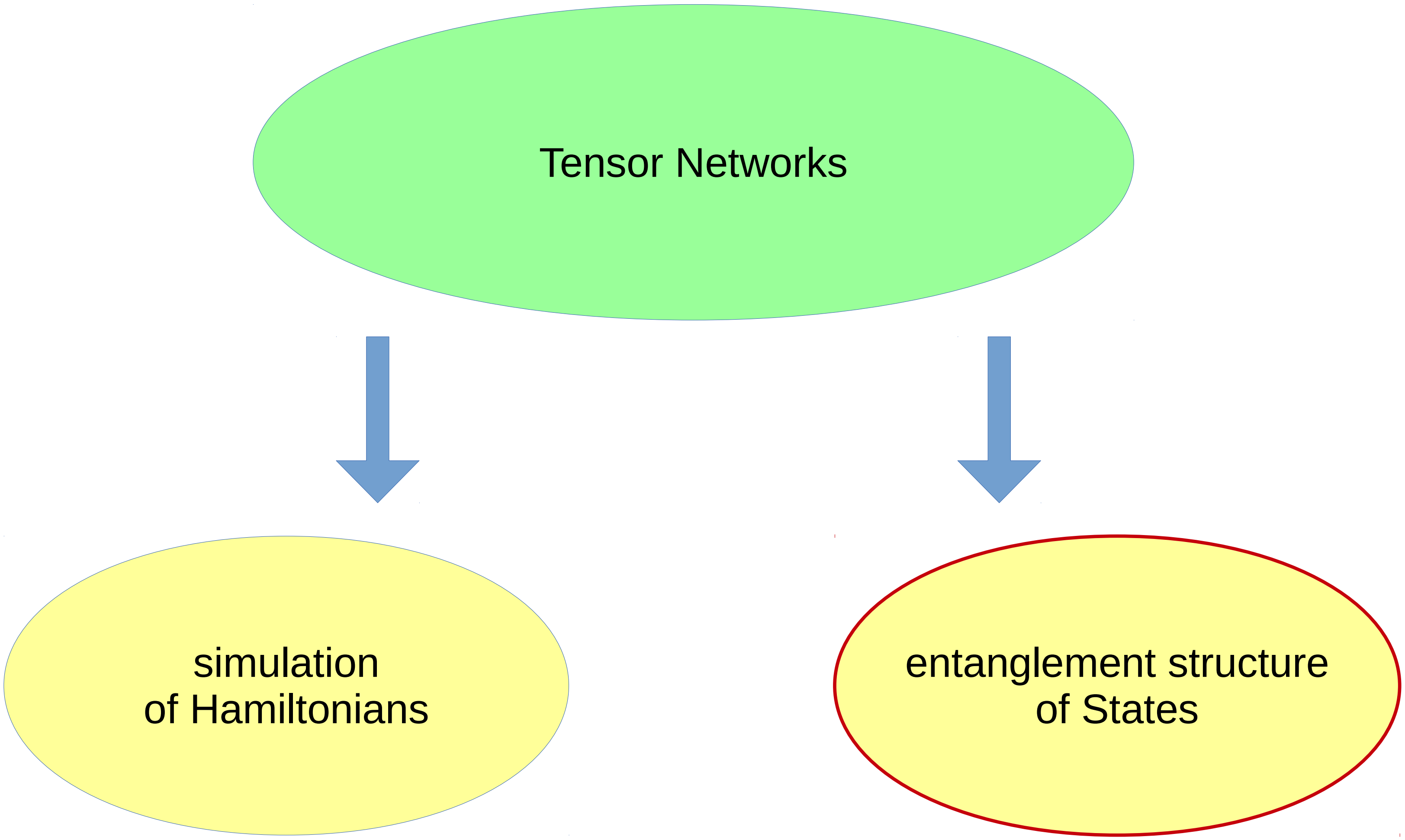
Max $\chi$ = 81

$\chi$=1 error=27

$\chi$=2 error=14.8

$\chi$=3 error=9.2

$\chi$=4 error=7.1


Tensor Networks
simulation
of Hamiltonians
entanglement structure
of States

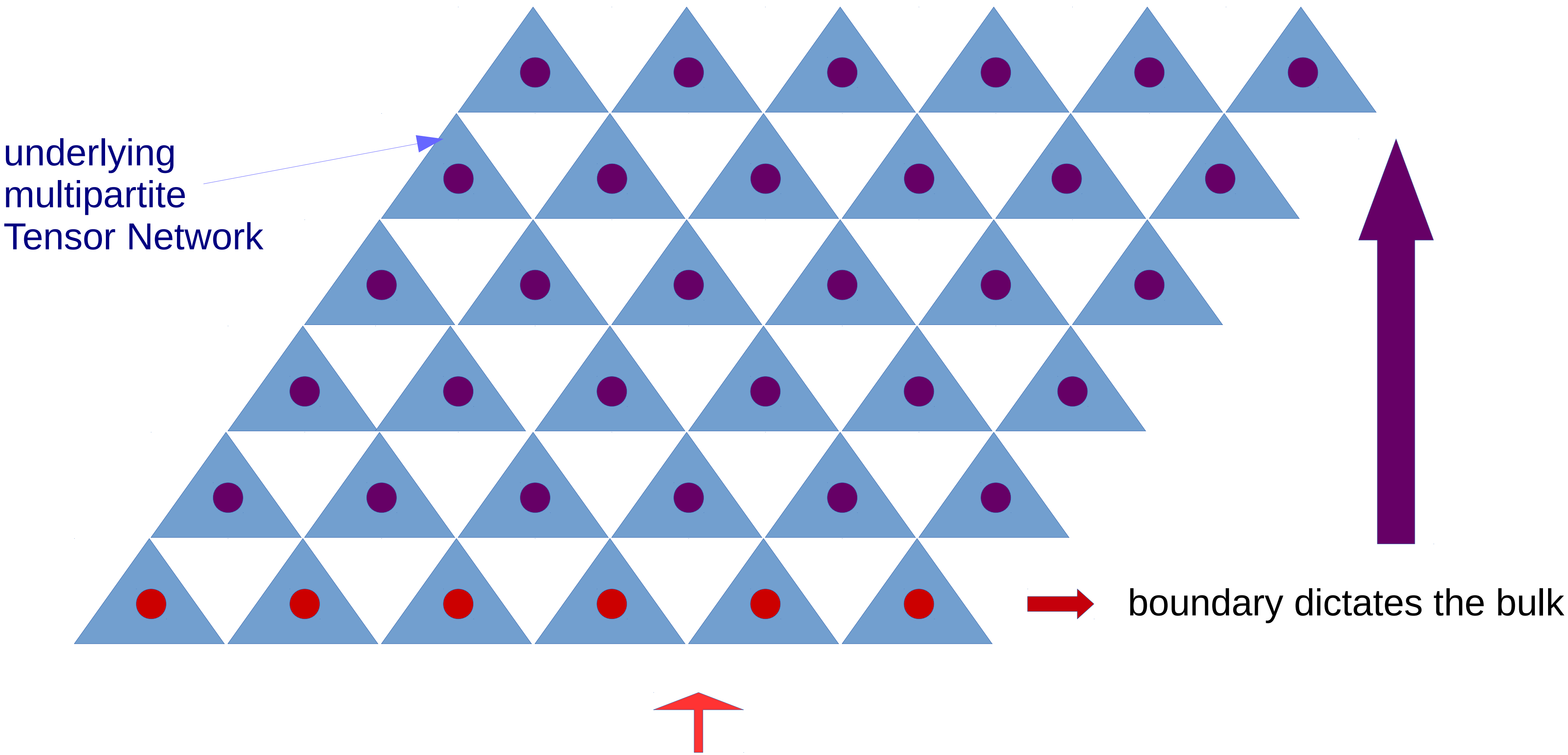
# H code

$$|H\rangle = \sum_i |i\rangle_{boundary} |\varphi_i\rangle_{bulk}$$

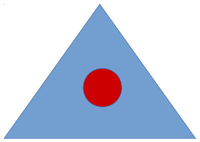
$$\langle i|j\rangle = \langle \varphi_i|\varphi_j\rangle = \delta_{ij}$$

$|i\rangle, |\varphi_i\rangle$ product states

# H code



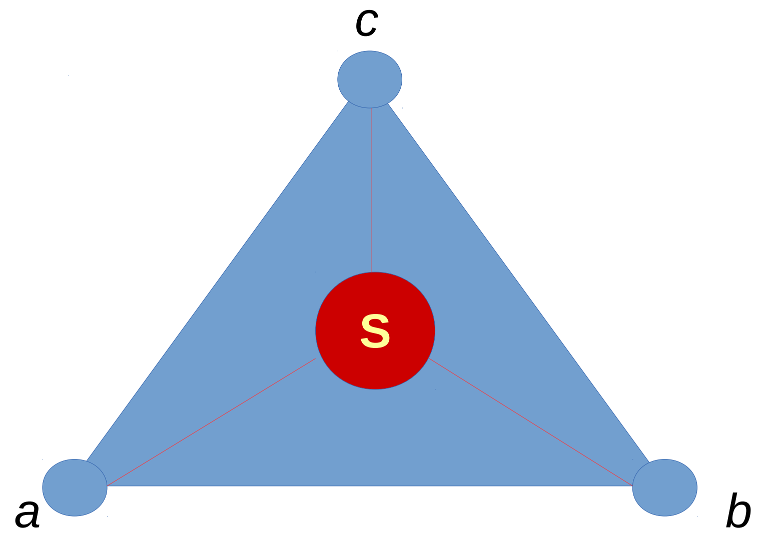

$$|H\rangle = \sum_i |i\rangle_{boundary} |\varphi_i\rangle_{bulk}$$



Choice of tensor network = choice of simplex + connection of simplices

Choice of simplex

$$|\psi\rangle_{ABCS} = \sum_{abcs} t_{abcs} |a\, b\, c\rangle_{ancillae} |s\rangle_{physical}$$

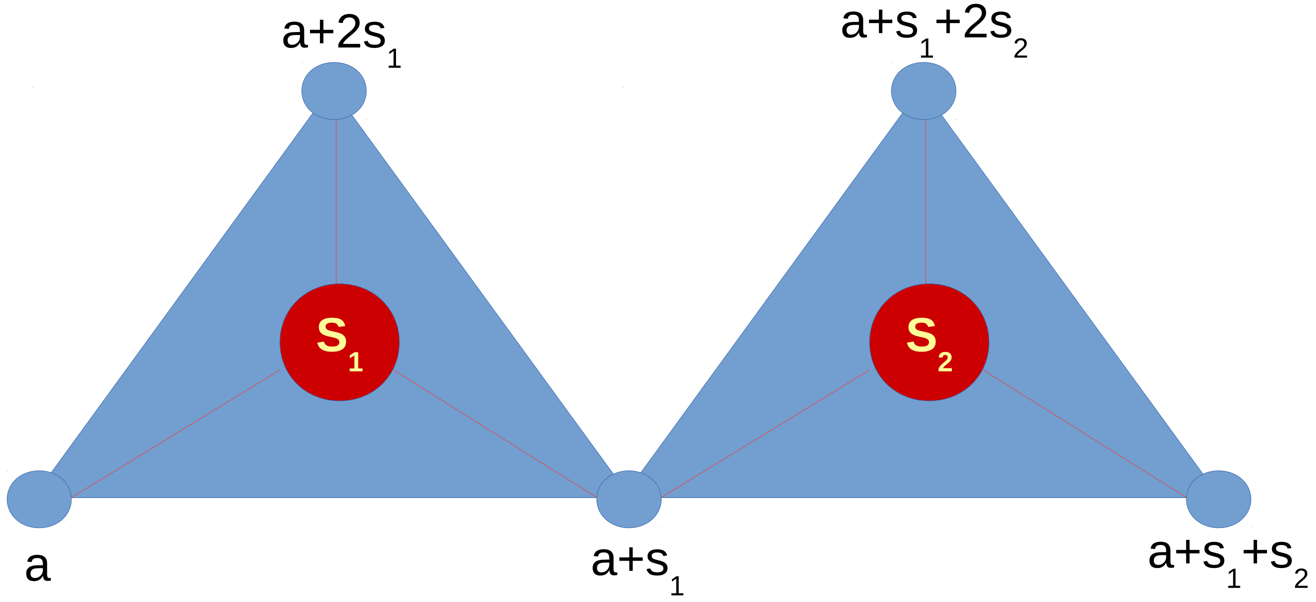



$$t_{abcs} = A^{s}_{abc}$$

Let's combine

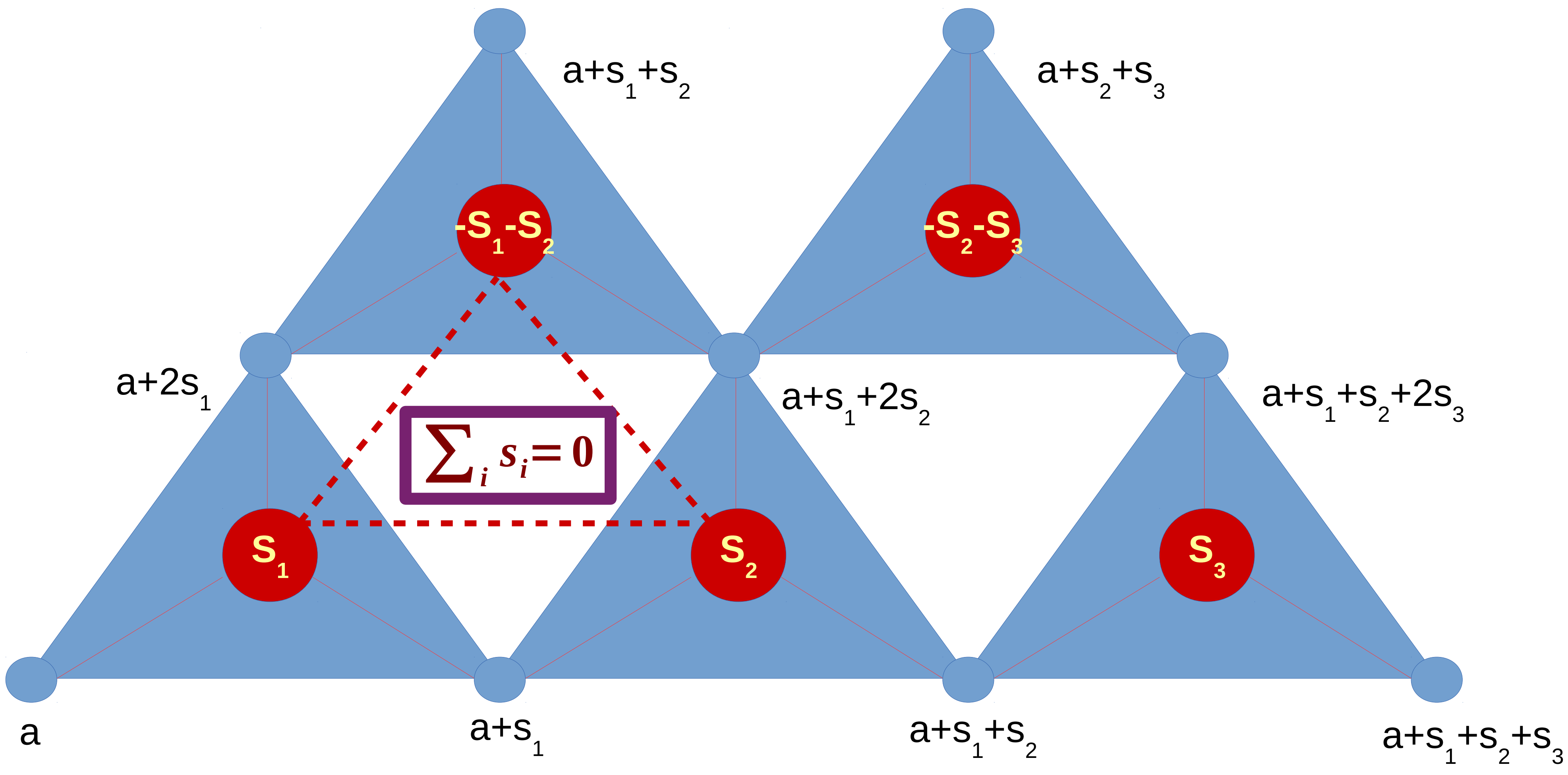
Tensor Network + AME

# Build network by linking ancillae



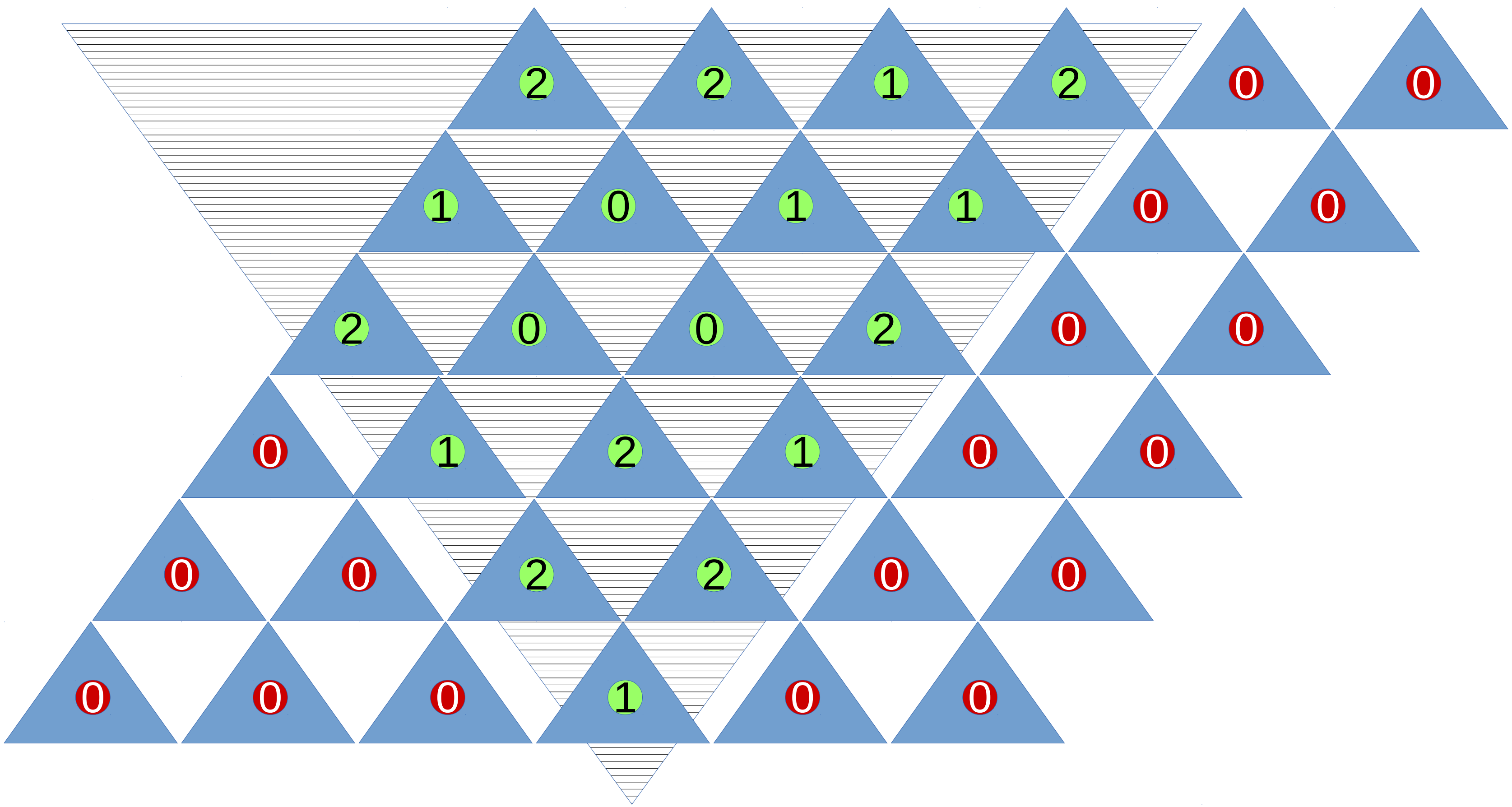

Ancillae are shared from one simplex to the contiguous ones

Emergence of **neutralization rule** = holography




Neutralization = Gauss law ?

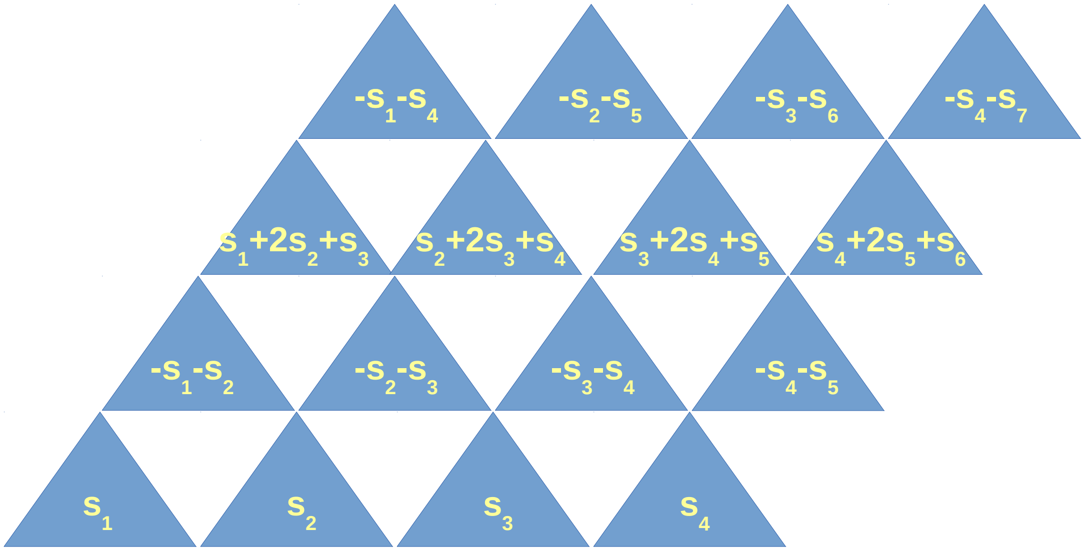
# Light-cone influence of a boundary qutrit



# States on a Torus

The neutralization rule may enter in conflict with periodicity

Yet, if $s_1=s_4$ then $-s_1-s_4=-2\,s_1=s_1$




Periodicity shows up for size
$n=m=3^k$

and for special sizes
$(n,m)=(5,40),(7,182),(11,121),\ldots$

# A proof of periodicity for *(n,m)=3^k*

$$\begin{pmatrix} s_1' \\ s_2' \\ \dots \\ s_n' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -1 & 0 & \dots & 0 \\ 0 & -1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & -1 \end{pmatrix}}_{T_n = -I - U_n} \begin{pmatrix} s_1 \\ s_2 \\ \dots \\ s_n \end{pmatrix}$$

$$U_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad U_n^n = I$$

$$T_n^n = (-I - U_n)^n (mod\ 3) = -\sum_{r=0}^{n} \binom{n}{r} U_n^r (mod\ 3) = -(I + U_n^n)(mod\ 3) = -2\ I\ mod\ (3) = I$$

$$n = 3^k$$

(Some tori sizes only accept a subset of states)

# Topological properties of the 3x3 torus

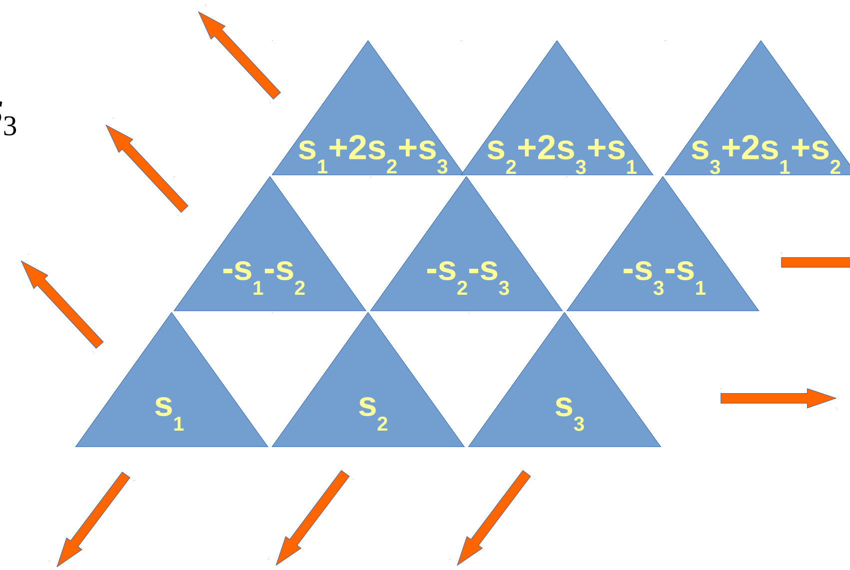
$$S = \sum_i s_i$$

$$Q = \prod_i z_i$$

$$z_i = q^{1+s_i} \qquad q = e^{\frac{2\pi i}{3}}$$

REDUNDANCY: **All cycles carry the same redundant information !**

$S = s_1 + s_2 + s_3$




$S = s_1 + s_2 + s_3$

$S = -2s_1 - 2s_2 - 2s_3 = s_1 + s_2 + s_3$

$S = s_1 + s_2 + s_3$

$S = s_1 + s_2 + s_3$

Same is valid for larger tori

# H code

$$|H_S\rangle = \sum_{s_i / \sum s_i = S} |s_i\rangle_{boundary} |f(s_i)\rangle_{bulk}$$

A qutrit can be encoded in $\left(|H_0\rangle, |H_1\rangle, |H_2\rangle\right)$

through the value of topological redundant observables

If Hamming distance is large, robust quantum memory is possible

Symmetry + Neutralization rule

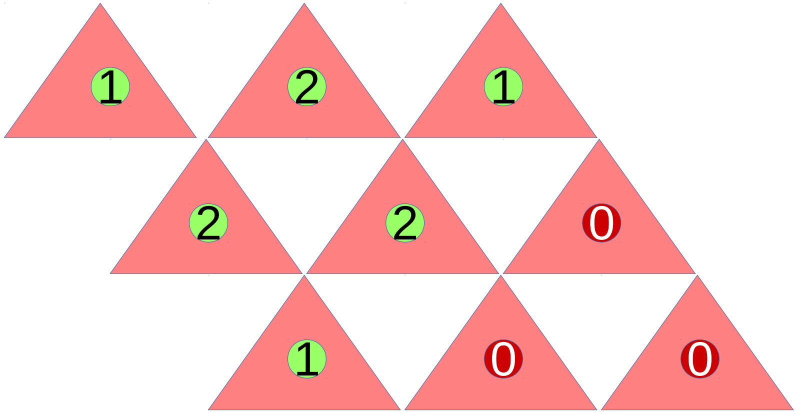
Any triangle will carry all possibilities equally weighted

…..   …..   …..   …..   …..   …..
0 ...   0 ...   1 ...   1 ...   2 ...   2 ...
1 2 s   2 1 s   0 2 s   2 0 s   1 0 s   0 1 s

…..   …..   …..
0 ...   1 ...   2 ...
0 0 s   1 1 s   2 2 s

**No local observable can distinguish the three phases**

Revisit the light-cone in *3x3* torus: large Hamming distance!




Hamming distance = *6*

For size $n=m=3^k$ Hamming distance = $6^k$

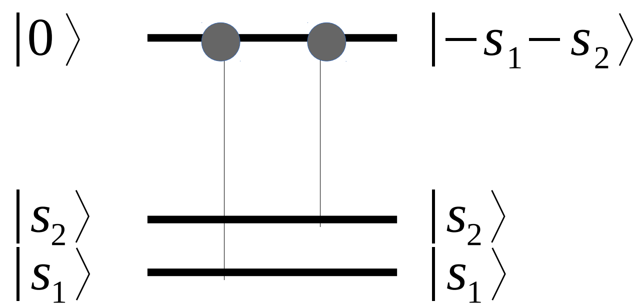
*dist(3x3)=6* *dist(9x9)=36*

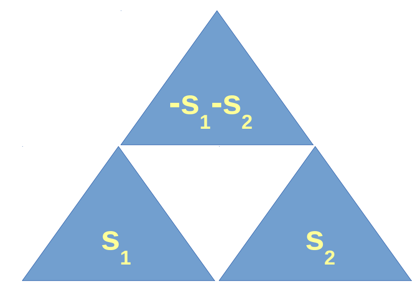
Fractal dimension

boundary=$n$ volum=$n^2$

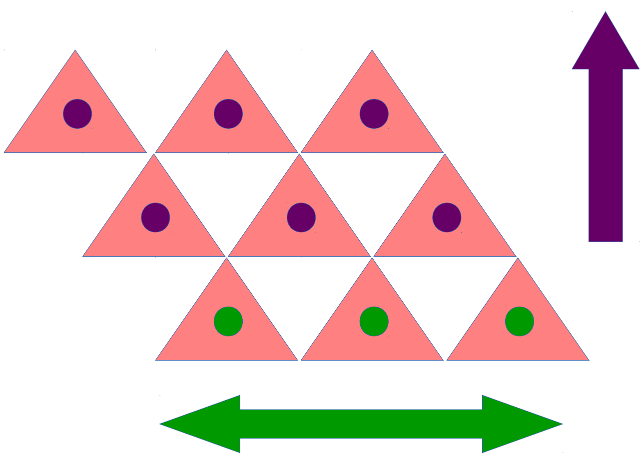
$dist=n^{(\log 6/\log 3)} = n^{1.63091}$

# Construction of a H-code as a cellular automata




bulk
sequencial
neutralization gates



$$U_{neutralization} = U_{1,3} U_{2,3}$$

$$U|a\rangle|b\rangle = |a\rangle|ba^2\rangle$$

$$a, b = 1, q, q^2$$

$$q = e^{\frac{2\pi i}{3}}$$

boundary
symmetrization

$$H_{boundary} = -\sum_{i=1}^{n} \sum_{a+b=0} X_i^a X_{i+1}^b$$

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

protects the total sum of the row

# Parent Hamiltonian

Trivial proof of non-locality of a parent Hamiltonian

$$\boldsymbol{H}|H\rangle = \boldsymbol{H}\left(|\psi_1\rangle + |\psi_2\rangle + \ldots\right)$$

dist = $6^k$

$$\boldsymbol{H}|\psi_1\rangle \sim |\psi_2\rangle$$

$\boldsymbol{H}$ must be $6^k$-local

$$\boldsymbol{H} = \boldsymbol{H}_{Z,neutralization} + \boldsymbol{H}_{X,link\ triangles} \qquad [\boldsymbol{H}_X, \boldsymbol{H}_Z] = 0$$

$$\boldsymbol{H}_{Z,neutralization} = \sum_{i,j,k \in \Delta} \left( 2 - Z_i Z_j Z_k - (Z_i Z_j Z_k)^2 \right) \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}$$

Produces a degenerate ground state, where every triangle is neutralized

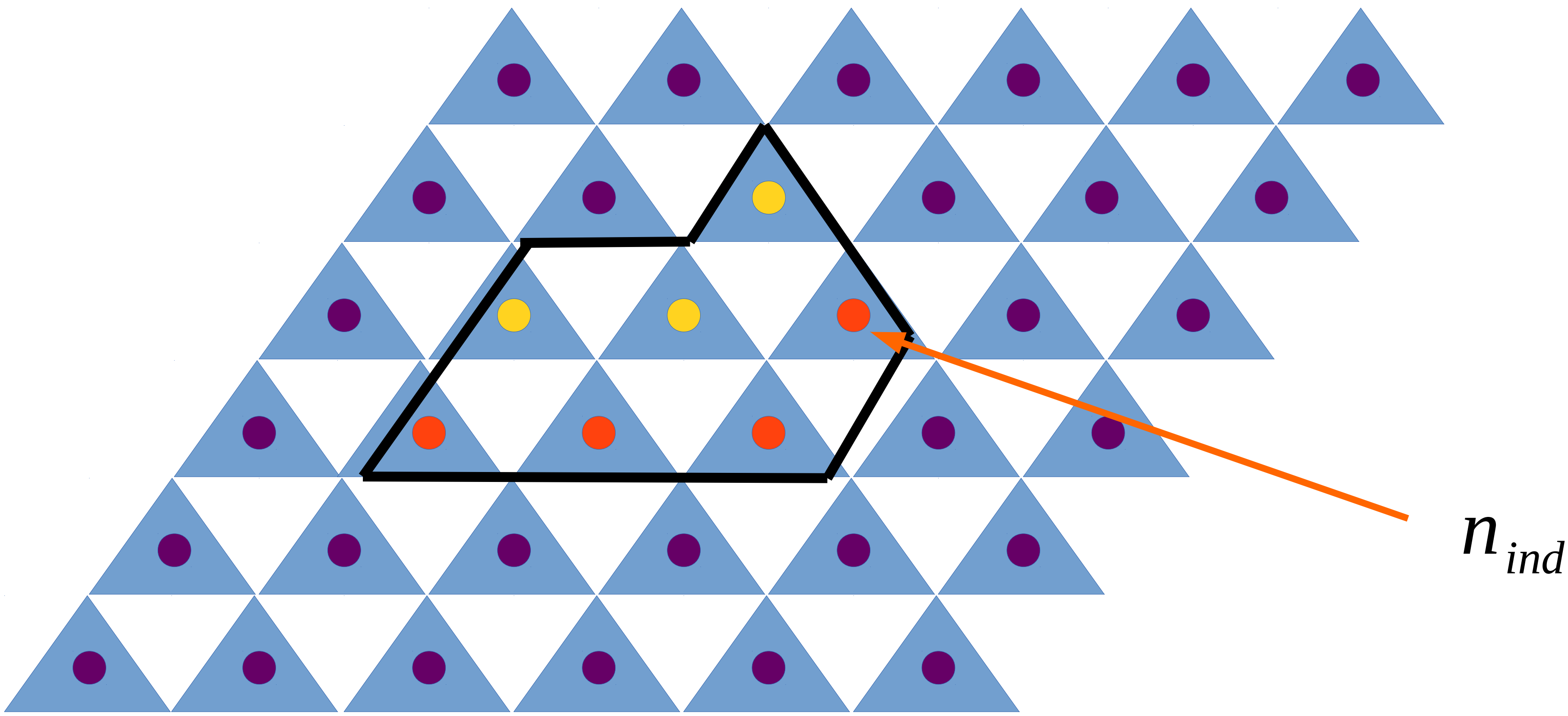
To break the degeneracy and obtain a single ground state, link triangles with

$$\boldsymbol{H}_{X,link\ triangles} = -\sum_{p,n=0,1,2} \prod_{a,b=1,\ldots,3^k} X_{a,b}^{-n(a-b+p)}$$

where *X* acts on the site *a,b*

***H*** *is not unique! The tensor network is not injective*

Entropy of entanglement




$$S = -Tr\,\rho_i \log_3 \rho_i = n_{ind}$$

$$S_{top} = S_{ABC} - S_{AB} - S_{AC} - S_{BC} + S_A + S_B + S_C = -1$$

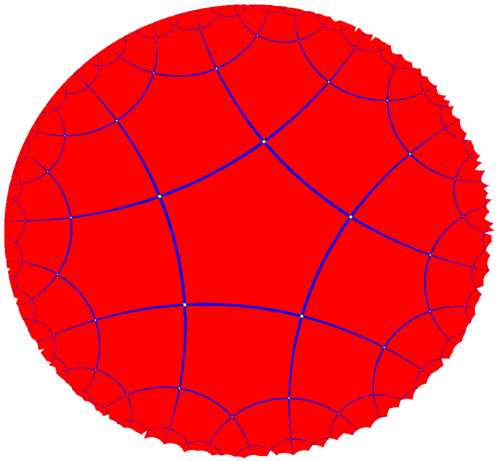
# Multi-Unitarity

AME iff

$$t_{abcd} = U_{(ab)(cd)} = U_{(ac)(bd)} = U_{(ad)(bc)}$$

No sense of direction for RG interpretation
Emergence of symmetry and holography

Is it necessary? K-uniformity?



Isometries: Pastawski, Yoshida, Harlow, Preskill: AME(5,2)
Combinatorial design: Goyeneche, Alsina, JIL, Życzkowski

# The most entangled spin chain

CFT

Entropy is maximal at criticality

$$S \sim \frac{c}{3}\log n + const$$

QMA

2-local translational invariant is QMA

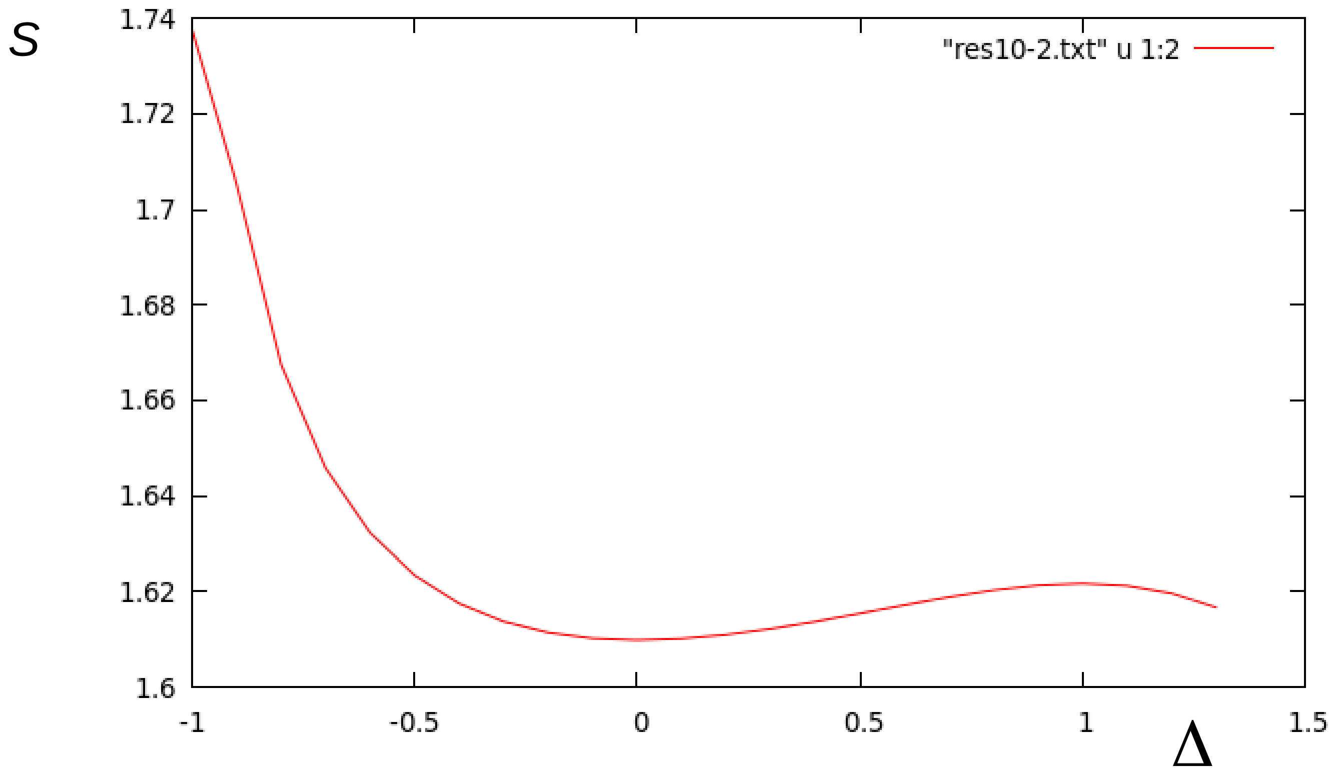
A spin 2 model proposed by Movassagh-Shor

$$S \sim \sqrt{n}$$

Minimal spin where QMA takes over?

Spin ½: numerical evidence sets max entanglement along the line

$$H=\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta\, \sigma_i^z \sigma_{i+1}^z\right)$$




Ferromagnetic Heisenberg:

$$\Delta \rightarrow -1^{+}$$

$$S=\frac{1}{2}\log(n+1)$$

No CFT !

Frustration beats conformal symmetry!

Preliminary results for higher spins

$$H = \sum_{a \notin Casimir} \lambda_i^a \lambda_{i+1}^a + \Delta \sum_{a \in Casimir} \lambda_i^b \lambda_{i+1}^b$$

Maximum at $\Delta \rightarrow -1^+$

| | |
|---|---|
| Spin 1/2 | Log n |
| Spin 1 | Log $n^2$ |
| Spin 3/2 | Log $n^3$ |

No sign of QMA yet

# Gauge theory from MaxEnt

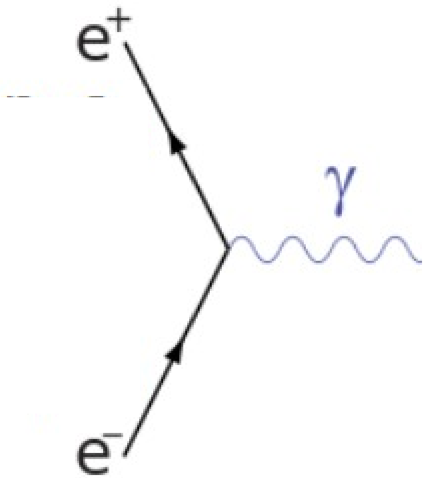
Conformal symmetry is obtained as a (local) max of entropy in 1D

Could we think of gauge symmetry as a consequence of maximal entanglement

**MaxEnt**

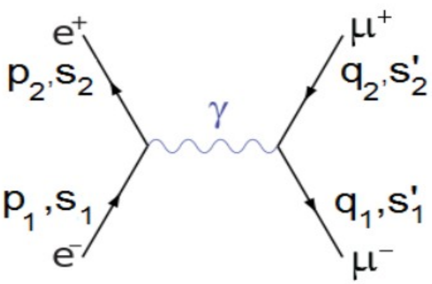
?

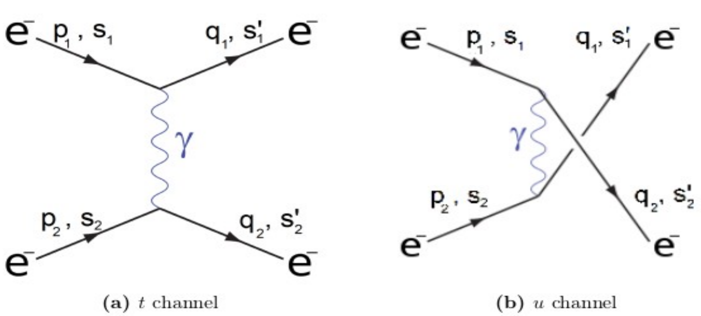
Let the QED vertex free




$$e\,\gamma^{\mu} \rightarrow e\,G^{\mu}$$

And fix $G^{\mu}$ demanding max entanglement in processes




.......

1) Impose hermiticity, parity, charge conjugation and time reversal

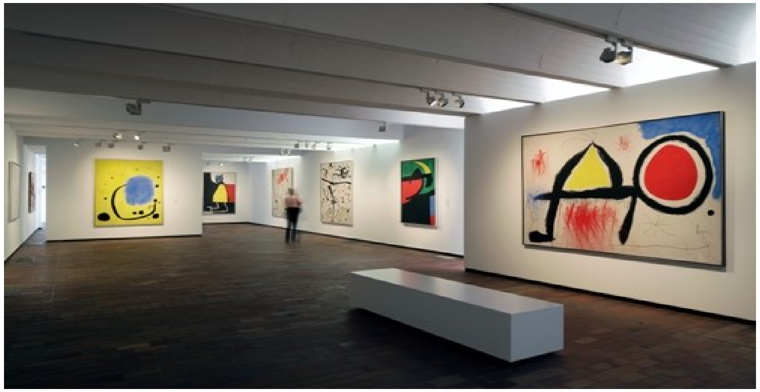
2) Impose generation of MaxEnt and conservation of MaxEnt
for Moller, Bhabha, Mott scatterings

$$G^0 = \begin{pmatrix} 0 & \pm I \\ \mp I & 0 \end{pmatrix} \qquad G^i = \begin{pmatrix} 0 & \pm \sigma^i \\ \mp \sigma^i & 0 \end{pmatrix}$$

Further signs may be constrained by Compton scattering

# CONCLUSION

There is life in the MaxEnt part of the Hilbert space



# THANKS

see you in BCN

TNS ancillary maximal entanglement generates interesting states

Absolute Maximally Entangled 4-qutrit state generates a holographic code

3-fold degenerate ground state, large hamming distance, non-local observables

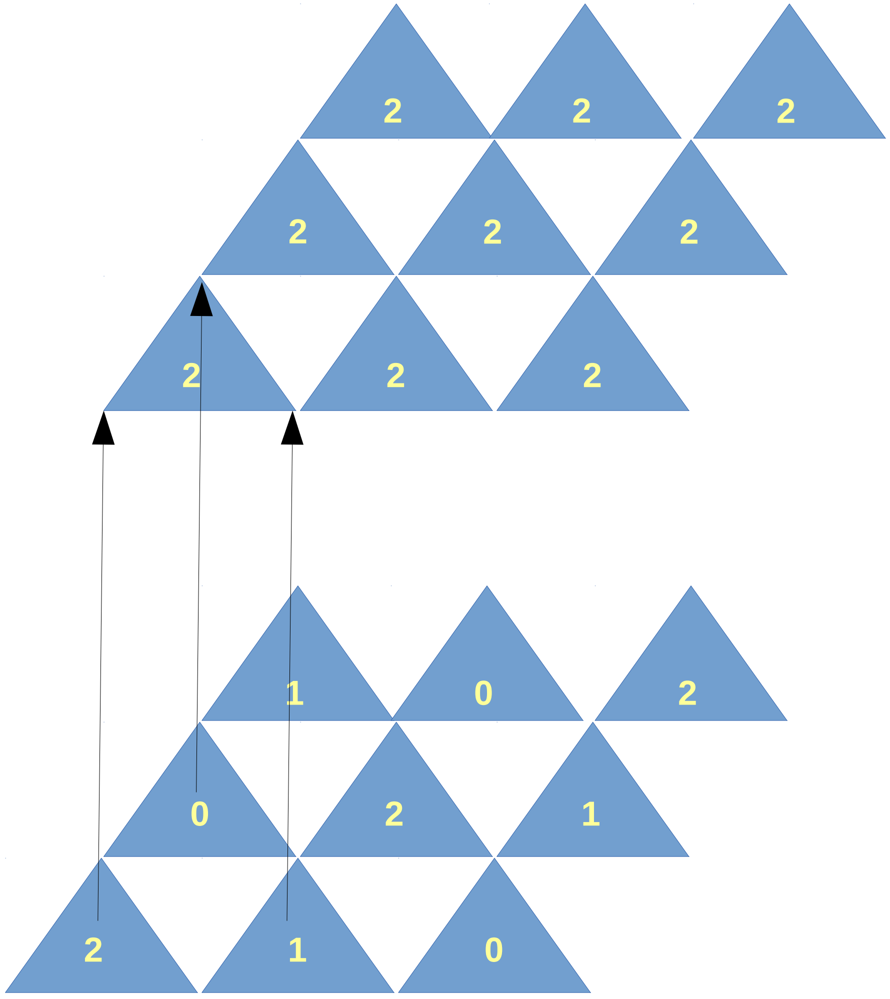
Holography = Multi-Unitarity ?

Gauss law is coming from gauge symmetry
(Celi, Tagliacozzo, Sierra, JIL)

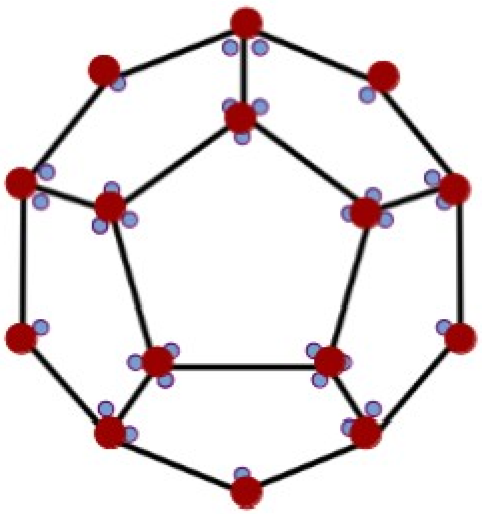
# 3D ?

2 non-trivial layers + fixed point

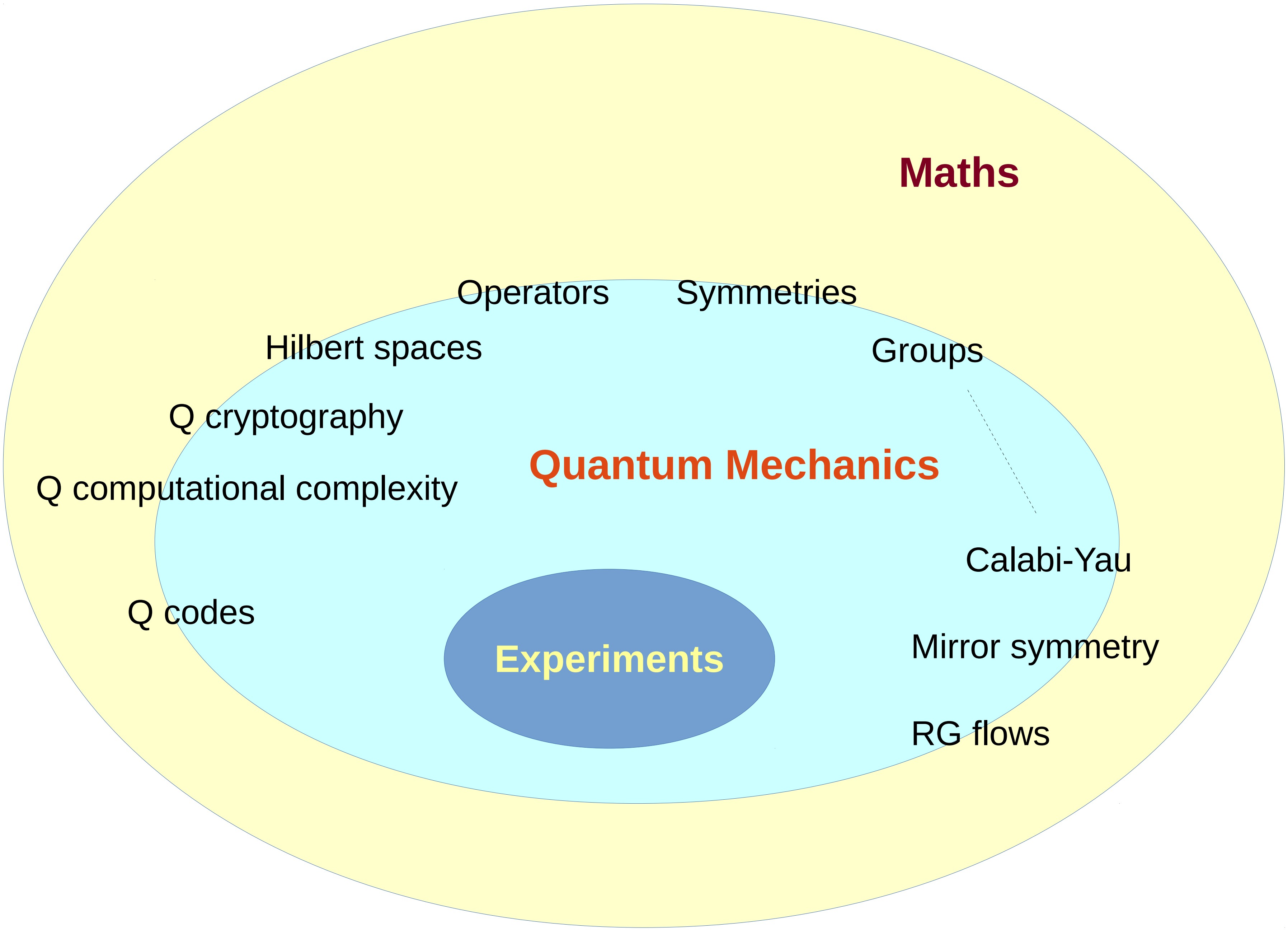
| *i* | *j* | *i+j* | *i+2j* |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 2 |
| 0 | 2 | 2 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 2 | 0 |
| 1 | 2 | 0 | 2 |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 2 | 2 | 1 | 0 |



# Divertimento on platonic entanglement



| | | |
|---|---|---|
| Tetrahedron | Faces = 4 | AME(4,3) |
| Tetrahedron | Edges = 6 | AME(6,5) |
| Tetrahedron | Vertices = 4 | AME(4,3) |
| Exahedron | Faces = 6 | AME(6,5) |
| Exahedron | Edges = 12 | AME(12,11) |
| Exrahedron | Vertices = 8 | AME(8,7) |
| Octahedron | Faces = 8 | AME(8,7) |
| Octahedron | Edges = 12 | AME(12,11) |
| Octahedron | Vertices = 6 | AME(6,5) |
| Dodecahedron | Faces = 12 | AME(12,11) |
| Dodecahedron | Edges = 30 | AME(30,29) |
| Dodecahedron | Vertices = 20 | AME(20,19) |
| Icosahedron | Faces = 20 | AME(20,19) |
| Icosahedron | Edges = 30 | AME(30,29) |
| Icosahedron | Vertices = 12 | AME(12,11) |



Maths
Operators
Symmetries
Hilbert spaces
Groups
Q cryptography
Quantum Mechanics
Q computational complexity
Calabi-Yau
Q codes
Experiments
Mirror symmetry
RG flows

# Why Quantum Mechanics?

Superposition $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

Entanglement $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

Observables: Hermitian operators

$$O|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$$

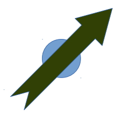
Measurements: Eigenvalues of Observables

Probability: $|\langle\psi_\lambda|\psi\rangle|^2$

Evolution: $|\psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)H}|\psi(t_0)\rangle$

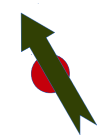
# Falsifiability of Quantum Mechanics: Bell inequalities

Alice



$$P(\vec{A}, a; \vec{B}, b)$$

Bob



$\vec{A} = \vec{n}_A \cdot \vec{\sigma}$ measurement of spin in random directions

$a$ result of the measurement, take values +1 or -1

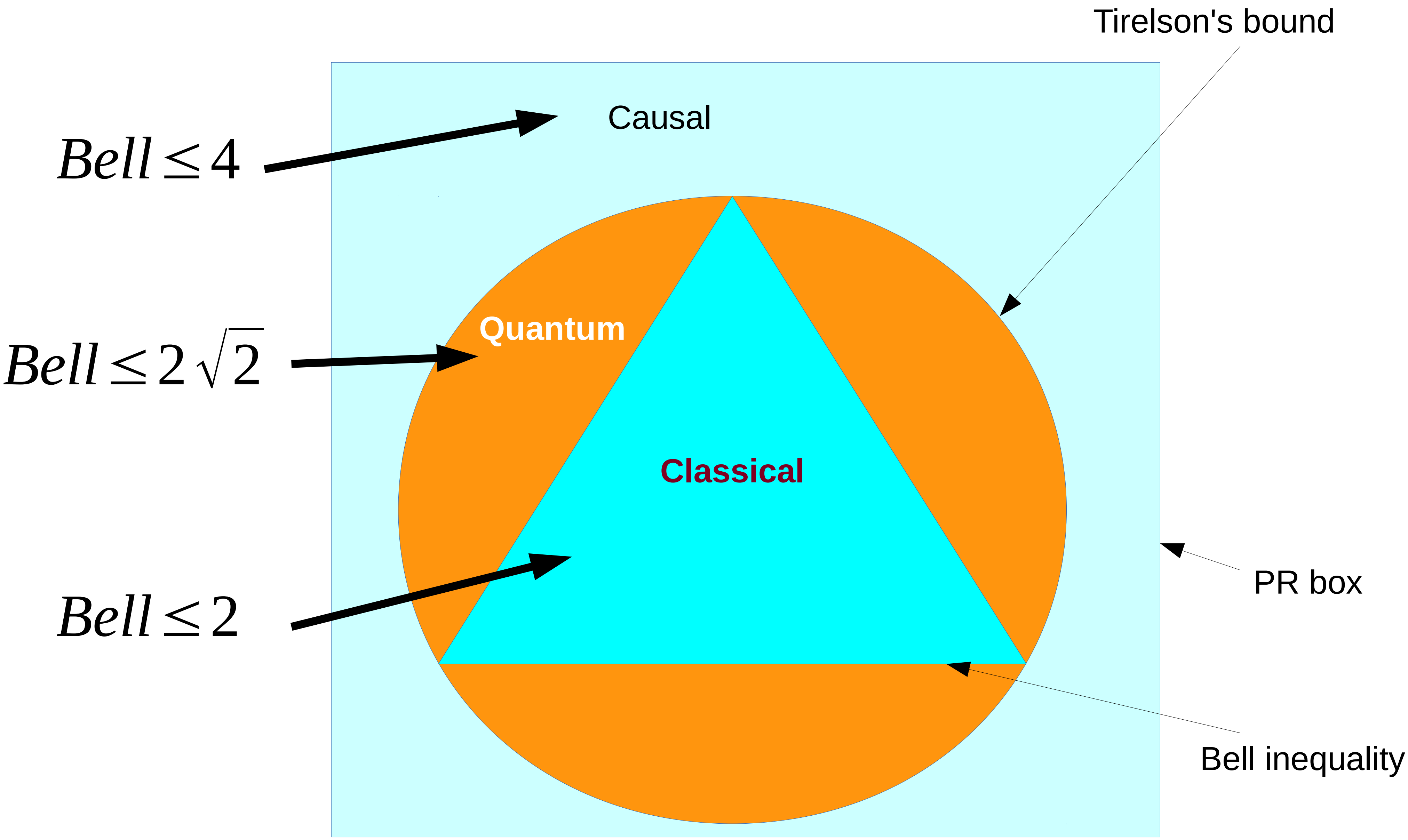
Measurements deliver a **probability distributions**

$$Bell = \left| \langle \vec{A}_1 \vec{B}_1 + \vec{A}_1 \vec{B}_2 + \vec{A}_2 \vec{B}_1 - \vec{A}_2 \vec{B}_2 \rangle \right|$$

In Classical Physics $\quad Bell \leq 2$

Einstein: element of reality, God does not play dice in the atoms

## Classical/Quantum/Causal polytopes for probabilities distributions




NATURE: BELL < 2.82

RG: *H* coarse grains in *H'*

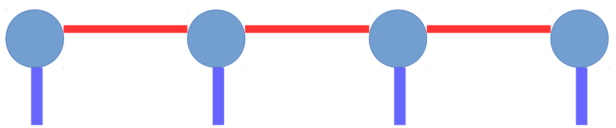
*H** is a fixed point of the flow

RG on states

(Verstraete, Cirac, JIL, Rico, Wolf)

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_n} c^{i_1, i_2, \ldots, i_n} |i_1, i_2, \ldots, i_n\rangle$$

Matrix Product State (MPS)



$$A^{i_1}_{\alpha\beta} A^{i_2}_{\beta\gamma} A^{i_3}_{\gamma\delta} A^{i_4}_{\delta\epsilon}$$

$$A^{i_1}_{\alpha\beta} A^{i_2}_{\beta\gamma} = \sum_{\lambda} U^{i_1 i_2}_{a} \lambda^{a} V^{a}_{\alpha\gamma}$$

Coarse-graining

$$\lambda^{a} V^{a}_{\alpha\gamma} = \bar{A}^{i_a}_{\alpha\gamma}$$

# Goal of this talk

Produce some Quantum Information example of Holography

**Operators** in the **Bulk**
dictated by
operators on its **Boundary**

**State** in the **Bulk**
dictated by
its value on the **Boundary**