Entanglement quantification made easy

Polynomial measures invariant under convex decomposition

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BR and G. Adesso, Phys. Rev. Lett. **116**, 070504 (2016)

Overview

- **How to quantify entanglement?**
  Introduction to entanglement measures and related concepts
- **What are the problems associated with it?**
  Mixed states and the convex roof extension
- **Can we simplify them in certain cases?**
  The one-root condition and measures invariant under convex decomposition
- **What are these cases useful for?**
  Applications in 2- and 3-qubit states as well as in monogamy relations

Quantifying entanglement

Entanglement measures

$E(|\psi\rangle)$ is an entanglement measure if

- $E(|\psi\rangle) = 0$ if $|\psi\rangle$ is separable
- E is an **entanglement monotone**:
  does not increase on average under (Stochastic) Local Operations and Classical Communication (SLOCC)
- E is convex:

$$E\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) \leq \sum_i p_i E\left(|\psi_i\rangle\right)$$

Extension to mixed states: the **convex roof**

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle) \quad \text{with} \quad \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$$

A. Uhlmann, Open Sys. & Inf. Dyn. **5**, 209 (1998)

G. Vidal, J. Mod. Opt. **47**, 355 (2000)

Polynomial measures

Polynomial invariant of homogeneous degree d:

$$P\big(c\,L\,|\psi\rangle\,\big) = c^{\,d}\,P\big(\,|\psi\rangle\,\big)$$

with $L \in SL(2,\mathbb{C})^{\otimes n}$, $c \in \mathbb{R}$, $|\psi\rangle \in \mathcal{H}_2^n$.

- iff $d \leq 4$, P is an entanglement monotone

  C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, PRA **85**, 022301 (2012)

- iff $d = 2$, $P\big(L\,\rho\,L^\dagger\big) = P\,(\rho)$

  O. Viehmann, C. Eltschka, and J. Siewert, Appl. Phys. B **106**, 533 (2012)

Examples of polynomial measures:

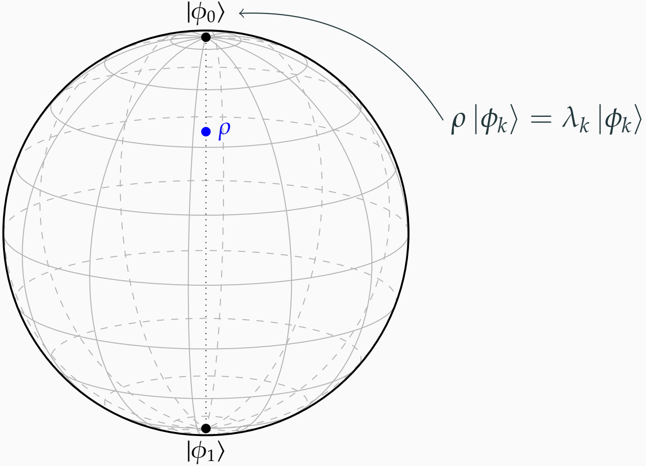
- concurrence C ($n = 2$, $d = 2$)

  S. Hill and W. K. Wootters, PRL **78**, 5022 (1997)

- three-tangle τ ($n = 3$, $d = 4$)

  V. Coffman, J. Kundu, and W. K. Wootters, PRA **61**, 052306 (2000)

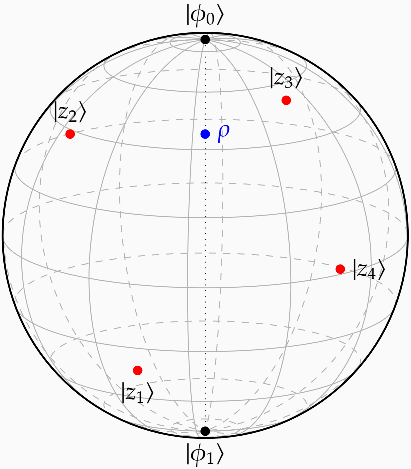
Zero polytope



$$E(|z\rangle) = E\Big(\sum_{j=0}^{\eta-1} \omega_j\,|\phi_j\rangle \Big) = 0$$

rank 2

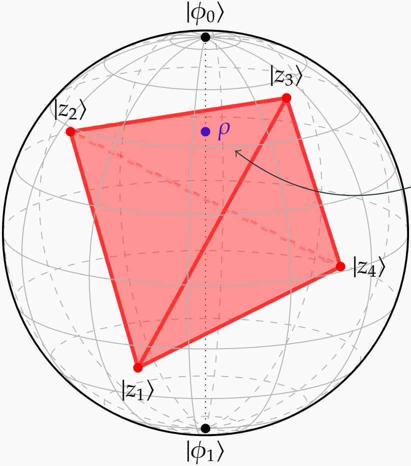
Zero polytope



$$E(|z\rangle) = E\left(\sum_{j=0}^{\eta-1} \omega_j |\phi_j\rangle \right) = 0$$

degree 4 — rank 2

Zero polytope



$\rho \in$ zero polytope
$\Rightarrow E(\rho) = 0$

$$E(|z\rangle) = E\left(\sum_{j=0}^{\eta-1} \omega_j \, |\phi_j\rangle \right) = 0$$

degree 4 rank 2

Zero polytope

R. Lohmayer et al.,

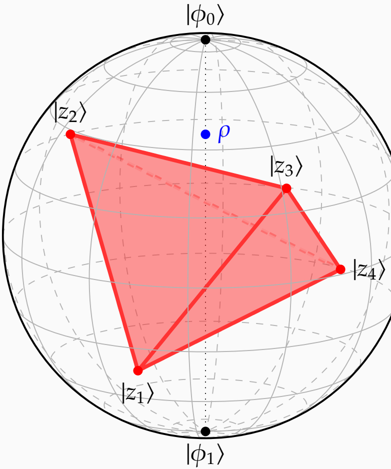
PRL **97**, 260502 (2006)

A. Osterloh et al.

PRA **77**, 032310 (2008)

S. Rodriques et al.

PRA **90**, 012340 (2014)



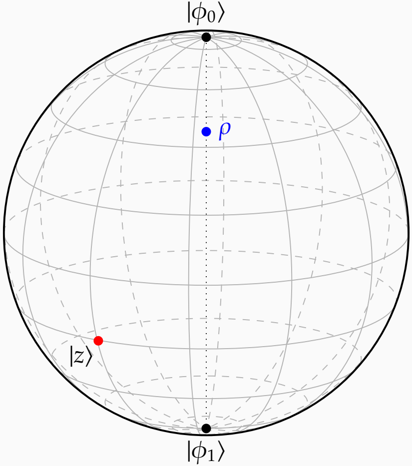
$\rho \notin$ zero polytope

$\Rightarrow$ may be entangled

$$E(|z\rangle) = E\Big(\sum_{j=0}^{\eta-1} \omega_j |\phi_j\rangle \Big) = 0$$

degree 4 — rank 2

Zero polytope



ρ is *one-root*

$$E(|z\rangle) = E\Big(\sum_{j=0}^{\eta-1} \omega_j \, |\phi_j\rangle \Big) = 0$$

rank 2

degree 4

Zero polytope

$$\underbrace{E\left(\sum_{j=0}^{\eta-1} \omega_j \left|\phi_j\right\rangle\right)}_{|P(\omega_1,\omega_2,\ldots,\omega_\eta)|}$$

$\eta = 2$

$$\underbrace{E\left(\left|\phi_0\right\rangle + \omega \left|\phi_1\right\rangle\right)}_{|P(\omega)|}$$

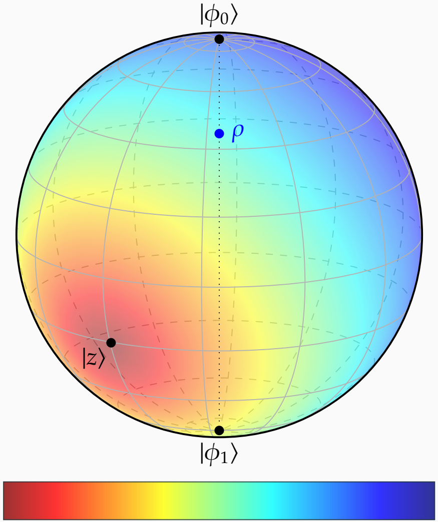
one-root

$$N\left|(\omega - z)^d\right|$$

$d = 2$

$$N\left\|\boldsymbol{\omega} - \boldsymbol{z}\right\|^2$$

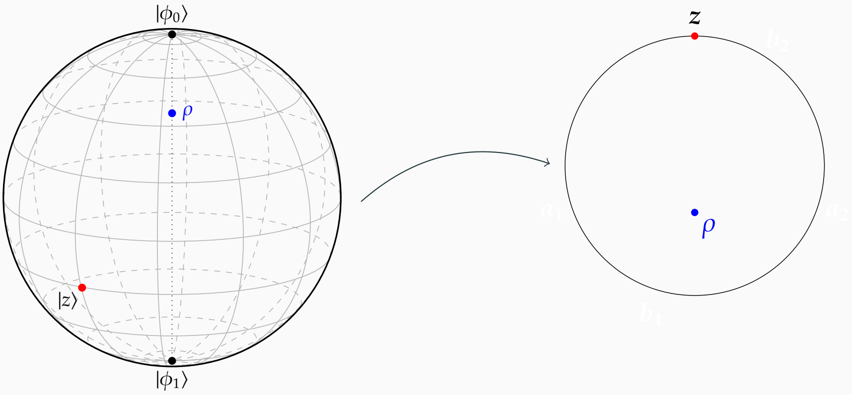
Graphical representation



smaller distance to $|z\rangle$
(smaller entanglement)

larger distance to $|z\rangle$
(larger entanglement)

Geometric properties of mixed states



Geometric properties of mixed states

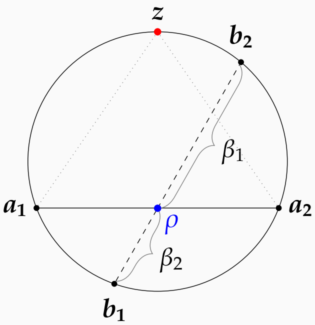
any decomposition:

$$\rho = \beta_1 |b_1\rangle \langle b_1| + \beta_2 |b_2\rangle \langle b_2|$$

equidistant decomposition:

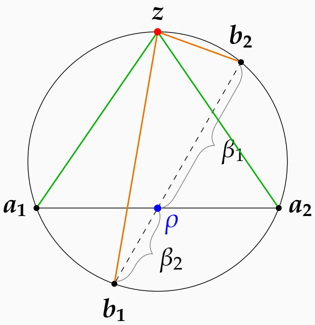
$$\rho = \alpha_1 |a_1\rangle \langle a_1| + \alpha_2 |a_2\rangle \langle a_2|$$

s.t. $\|\boldsymbol{z} - \boldsymbol{a}_1\| = \|\boldsymbol{z} - \boldsymbol{a}_2\|$



By Stewart's theorem

$$\alpha_1\|\boldsymbol{z}-\boldsymbol{a}_1\|^2+\alpha_2\|\boldsymbol{z}-\boldsymbol{a}_2\|^2=$$
$$=(\alpha_1+\alpha_2)\left(\|\boldsymbol{z}-\boldsymbol{\rho}\|^2+\alpha_1\alpha_2\right)$$
$$\Rightarrow\ \|\boldsymbol{z}-\boldsymbol{\rho}\|^2=\|\boldsymbol{z}-\boldsymbol{a}_1\|^2-\alpha_1\alpha_2$$



$$\beta_1\|\boldsymbol{z}-\boldsymbol{b}_1\|^2+\beta_2\|\boldsymbol{z}-\boldsymbol{b}_2\|^2=$$
$$=\left(\|\boldsymbol{z}-\boldsymbol{a}_1\|^2-\alpha_1\alpha_2+\beta_1\beta_2\right)$$

$$\beta_1\|\boldsymbol{z}-\boldsymbol{b}_1\|^2+\beta_2\|\boldsymbol{z}-\boldsymbol{b}_2\|^2=\|\boldsymbol{z}-\boldsymbol{a}_1\|^2$$

Geometric properties

Theorem.

Consider a finite set of points on an n-sphere $\{\alpha_i, \boldsymbol{a}_i\}$ equidistant from a chosen point $\boldsymbol{z}$ and denote by $\boldsymbol{g}$ their barycentre:

$$\boldsymbol{g} = \sum_i \alpha_i \boldsymbol{a}_i.$$

Then, for any finite set $\{\beta_j, \boldsymbol{b}_j\}$ with the same barycentre $\boldsymbol{g}$, the following holds:

$$\textstyle\sum_j \beta_j \|\boldsymbol{z} - \boldsymbol{b}_j\|^2 = \|\boldsymbol{z} - \boldsymbol{a}_m\|^2, \quad \forall\ \boldsymbol{a}_m \in \{\boldsymbol{a}_i\}.$$

Geometric properties

Theorem.

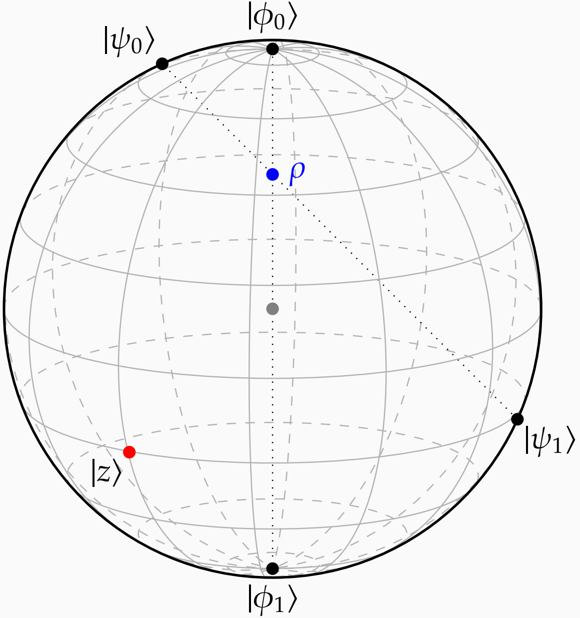
Consider a convex decomposition of a rank-2 one-root state ρ into the set $\{p_i, |\psi_i\rangle\}$ such that all Bloch vectors of $\{|\psi_i\rangle\}$ are equidistant from the zero-entanglement state $|z\rangle$.

Then, for any other decomposition of ρ into $\{s_j, |\sigma_j\rangle\}$, the following holds:

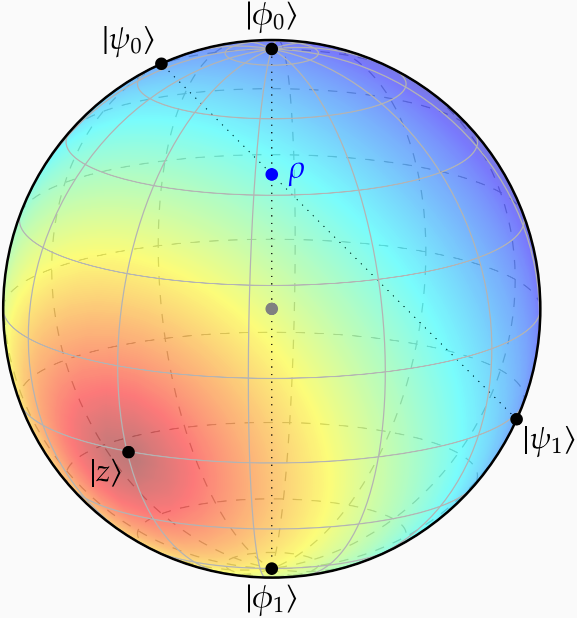
$$E\left(\sum_j s_j |\sigma_j\rangle \langle\sigma_j|\right) = E\left(|\psi_m\rangle\right), \quad \forall \ |\psi_m\rangle \in \{|\psi_i\rangle\}.$$

$$\Rightarrow \quad E(\rho) = E\left(\sum_j s_j |\sigma_j\rangle \langle\sigma_j|\right) = \sum_j s_j E\left(|\sigma_j\rangle\right)$$

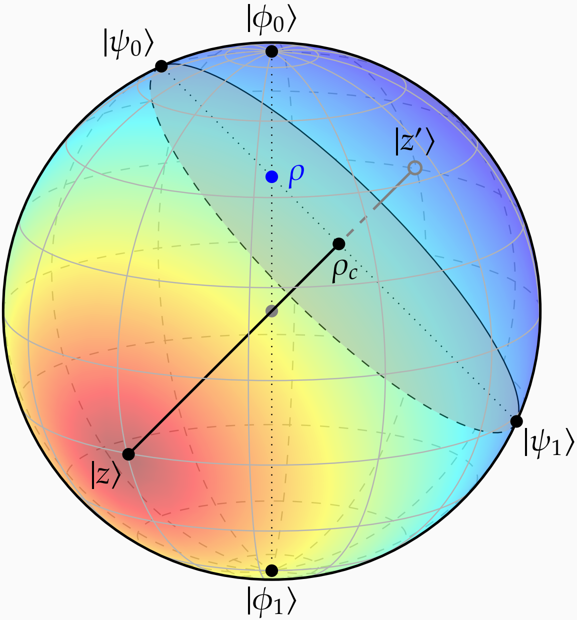
Graphical representation



Graphical representation



Graphical representation

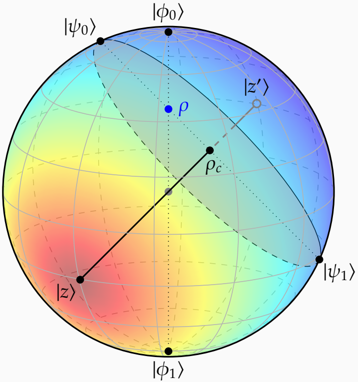


Examples and applications

Closed formula

$$\rho_c = \frac{1}{2}\|\boldsymbol{\rho_c} - \boldsymbol{z}\| \, |z'\rangle \langle z'| + \frac{1}{2}\|\boldsymbol{\rho_c} - \boldsymbol{z'}\| \, |z\rangle \langle z|$$

$$E(\rho) = E(\rho_c) = \frac{1}{2}\|\boldsymbol{\rho_c} - \boldsymbol{z}\| \, E\big(|z'\rangle \big) = D_{\mathrm{Tr}}\big(\rho_c, |z\rangle \langle z| \big) \, E\big(|z'\rangle \big)$$



Example: concurrence of 2 qubits

$$\rho = \sum_{i,j} \frac{1}{2} \left(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma} \right) |\phi_i\rangle \langle\phi_j|$$

$$\text{with } \boldsymbol{r} = r \left(\sin\theta \cos\varphi,\ \sin\theta \sin\varphi,\ \cos\theta \right)^T$$

$$|\phi_0\rangle = |z\rangle = |00\rangle$$

$$|\phi_1\rangle = |z'\rangle = \cos\frac{\gamma}{2} |01\rangle + \sin\frac{\gamma}{2} e^{i\delta} |10\rangle$$

$$C(\rho) = \frac{1}{2} \big| 1 - \langle\phi_0|\rho|\phi_0\rangle + \langle\phi_1|\rho|\phi_1\rangle \big| \; C(|\phi_1\rangle)$$

$$= \frac{1}{2} \left(1 - r\cos\theta \right) \sin\gamma$$

Example: three-tangle of 3 qubits

$$\rho = \sum_{i,j} \frac{1}{2} \left(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma} \right) |\phi_i\rangle \langle\phi_j|$$

$$\text{with } \boldsymbol{r} = r \left(\sin\theta \cos\varphi,\ \sin\theta \sin\varphi,\ \cos\theta \right)^T$$

$$|\phi_0\rangle = |z\rangle = a\,|001\rangle + b\,|010\rangle + c\,|100\rangle$$

$$|\phi_1\rangle = |z'\rangle = g\,|000\rangle + t_1\,|011\rangle + t_2\,|101\rangle + t_3\,|110\rangle + e^{i\gamma} h\,|111\rangle$$

S. Tamaryan, T.-C. Wei, and D.K. Park, PRA **80**, 052315 (2009)

$$\sqrt{\tau}\,(\rho) = \frac{1}{2} \left| 1 - \langle\phi_0|\rho|\phi_0\rangle + \langle\phi_1|\rho|\phi_1\rangle \right| \sqrt{\tau}\,(|\phi_1\rangle)$$

$$= \sqrt{\left| \frac{g t_1 t_2}{a^9} \right|} \left| \sqrt{c t_1} + \sqrt{b t_2} \right| |1 - r\cos\theta|$$

$$\times \left| a^4 + \left[\left(\sqrt{c t_1} + \sqrt{b t_2} \right)^4 + a^2 \left(g^2 + t_1^2 + t_2^2 \right) \right]^2 \right|$$

SLOCC classification of four qubits

$$\rho_{ABC} = \mathrm{Tr}_D \left|\Psi_{ABCD}\right\rangle \left\langle\Psi_{ABCD}\right|$$

Four qubits can be entangled in **nine** different ways.

F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, PRA **65**, 052112 (2002)

$$\left|\Psi_{ABCD}\right\rangle = L\left|G_\mu\right\rangle \qquad \text{with } \mu \in \{1,\ldots,9\},$$

$$\rho_{ABC} = \mathrm{Tr}_D\left[L\left|G_\mu\right\rangle\!\left\langle G_\mu\right| L^\dagger\right] \qquad L \in \mathrm{SL}(2,\mathbb{C})^{\otimes 4}$$

$\left\|G_1\right\rangle_{a,b,c,d}$	✗	
$\left\|G_2\right\rangle_{a,b,c}$	✗	
$\left\|G_3\right\rangle_{a,b}$	✗	
$\left\|G_4\right\rangle_{a,b}$	✓	**for any qubit**
$\left\|G_5\right\rangle_{a}$	✓	for qubits 2 and 4
$\left\|G_6\right\rangle_{a}$	✗	
$\left\|G_7\right\rangle$	✓	for qubits 2, 3, and 4
$\left\|G_8\right\rangle$	✓	for qubits 2, 3, and 4
$\left\|G_9\right\rangle$	✗	

Monogamy relations

Conventional monogamy (CKW)

V. Coffman, J. Kundu, and W.K. Wootters, PRA **61**, 052306 (2000)

$$\tau_{1|2\,3} \geq \tau_{1|2} + \tau_{1|3}$$

Generalised monogamy

T. J. Osborne and F. Verstraete, PRL **96**, 220503 (2006)

$$\tau_{1|2\cdots n} \geq \sum_{j=2}^{n} \tau_{1|j}$$

Strong monogamy

BR, S. Di Martino, S. Lee, and G. Adesso, PRL **113**, 110501 (2014)

$$\tau_{1|2\cdots n} \overset{?}{\geq} \sum_{j=2}^{n} \tau_{1|j} + \sum_{\substack{j=2 \\ k>j}}^{n} \tau_{1|j|k} + \ldots + \sum_{j=2}^{n} \tau_{1|\cdots|j-1|j+1|\cdots|n}$$

Strong monogamy for 4 qubits

$$\tau_{1|234} \not\geq \tau_{1|2} + \tau_{1|3} + \tau_{1|4} + \tau_{1|2|3} + \tau_{1|2|4} + \tau_{1|3|4}$$

only class $|G_4\rangle$ fails

$$\tau(\rho) = \left[\min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi_i\rangle)} \right]^2$$

Fixes:

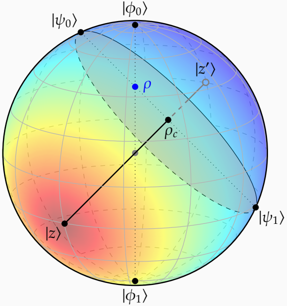
$$\tau(\rho) = \left[\min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi_i\rangle)} \right]^3$$

$$\tau(\rho) = \left[\min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \, \tau(|\psi_i\rangle)^{1/4} \right]^4$$

BR, A. Osterloh, and G. Adesso, in preparation

Conclusions

- Polynomial entanglement measures of degree 2 **do not depend on the chosen decomposition** for rank-2 states with only one unentangled state in their range (one-root states)
- One-root states occur in the marginals of several classes of four-qubit states



Further questions

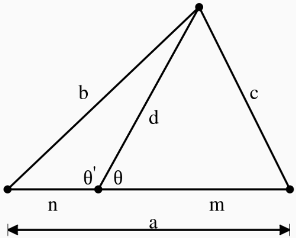
- How does the one-root condition generalise to states with higher rank?
- Are there classes of higher-dimensional systems that exhibit the one-root property?

Thank you

SLOCC classification

$$
\begin{aligned}
|G^1_{abcd}\rangle &= \tfrac{a+d}{2}(|0000\rangle + |1111\rangle) + \tfrac{a-d}{2}(|0011\rangle + |1100\rangle) \\
&\quad + \tfrac{b+c}{2}(|0101\rangle + |1010\rangle) + \tfrac{b-c}{2}(|0110\rangle + |1001\rangle) \\
|G^2_{abc}\rangle &= \tfrac{a+b}{2}(|0000\rangle + |1111\rangle) + \tfrac{a-b}{2}(|0011\rangle + |1100\rangle) \\
&\quad + c(|0101\rangle + |1010\rangle) + |0110\rangle \\
|G^3_{ab}\rangle &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) \\
&\quad + |0110\rangle + |0011\rangle \\
|G^4_{ab}\rangle &= a(|0000\rangle + |1111\rangle) + \tfrac{a+b}{2}(|0101\rangle + |1010\rangle) \\
&\quad + \tfrac{a-b}{2}(|0110\rangle + |1001\rangle) \\
&\quad + \tfrac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle) \\
|G^5_{a}\rangle &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
&\quad + i\,|0001\rangle + |0110\rangle - i\,|1011\rangle \\
|G^6_{a}\rangle &= a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle \\
|G^7\rangle &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle \\
|G^8\rangle &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle \\
|G^9\rangle &= |0000\rangle + |0111\rangle
\end{aligned}
$$

Stewart's theorem and Apollonius's formula



$$b^2m + c^2n = (m+n)(d^2 + mn)$$

Apollonius's formula:

$$\sum_i \alpha_i \|\mathbf{z} - \mathbf{a}_i\|^2 = \|\mathbf{z} - \mathbf{g}\|^2 + \sum_i \alpha_i \|\mathbf{g} - \mathbf{a}_i\|^2$$