Beyond SICs and MUBs: The conical designs

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Outline

Definition

A **spherical t-design** is a set of n normalized vectors $\{\phi_k\}_{k=1}^n \subset S^d$ such that the average value of any t-th order polynomial f_t over the set $\{\phi_k\}_{k=1}^n$ is equal to the average of f_t over all normalized vectors:

$$\frac{1}{n}\sum_{k=1}^{n} f_t(\phi_k) = \int_{S^d} f_t(\phi)\mathrm{d}\phi.$$

Example ($d = 3$)

- 1-design: triangular bipyramid
- 2-design: tetrahedron
- 3-design: cube, octahedron
- 5-design: icosahedron, dodecahedron

Definition

A **complex projective t-design** is a set of n pure states $\{\rho_k\}_{k=1}^n \subset \mathcal{P}(\mathbb{C}^d)$ such that

$$\frac{1}{n^2}\sum_{k,l=1}^{n} f_t(\mathrm{tr}(\rho_k\rho_l) = \iint_{\mathcal{P}(\mathbb{C}^d)^2} f_t(\mathrm{tr}\rho\sigma)\mathrm{d}\mu_H(\rho)\mathrm{d}\mu_H(\sigma)$$

for any any t-th order polynomial f_t, where μ_H denotes the unique unitarily invariant probability measure on $\mathcal{P}(\mathbb{C}^d)$.

Equivalently:

$$\frac{1}{n}\sum_{k=1}^{n} \rho_k^{\otimes t} = \binom{d+t-1}{t}^{-1} P_s,$$

where P_s is the projection onto symmetric subspace of $(\mathbb{C}^d)^{\otimes t}$.

Note: Every rank-1 normalized POVM can be considered as a complex projective 1-design

Examples of complex projective 2-designs

SIC-POVMs

A **symmetric informationally complete POVM (SIC-POVM)** consists of d^2 subnormalized rank-one projections $\Pi_j = |\phi_j\rangle\langle\phi_j|/d$ such that:

$$\mathrm{tr}(\Pi_i^* \Pi_j) = \frac{|\langle\phi_i|\phi_j\rangle|^2}{d^2} = \frac{1}{d^2(d+1)} \text{ for } i \neq j.$$

The states $\{|\phi_j\rangle\langle\phi_j|\}_{j=1}^{d^2}$ constitute a complex projective 2-design.

MUBs

The complete set of **mutually unbiased bases (MUB)** is a set of $d+1$ orthonormal bases $\{e_i^j\}_{i=1,j=1}^{d,d+1}$ in $\mathbb{C}^d$ such that

$$|\langle e_i^j | e_k^l\rangle|^2 = 1/d, \quad \text{for} j \neq l.$$

The states $\{|e_i^j\rangle\langle e_i^j|\}_{i=1,j=1}^{d,d+1}$ constitute a complex projective 2-design.

Theorem (Graydon, Appleby (2016))

Let $\{A_1, \ldots, A_m\}$ be a family of positive-semidefinite operators on $\mathbb{C}^d$. Then the following statements are equivalent:

i. $\sum_{j=1}^{m} A_j \otimes A_j$ *commutes with* $U \otimes U$ *for every unitary* U.

ii. *For some* $k_s \geq k_a \geq 0$

$$\sum_{j=1}^{m} A_j \otimes A_j = k_s P_s + k_a P_a.$$

Moreover, the A_j *span the space of selfadjoint operators on* $\mathbb{C}^d$ *iff* $k_s > k_a$.

Definition

A conical 2-design is a family of non-zero positive-semidefinite operators $\{A_1, \ldots, A_m\}$ on $\mathbb{C}^d$ satisfying the equivalent conditions in the above theorem with $k_s > k_a$.

M.A. Graydon, D.M. Appleby, J. Phys.A **49**, 085301 (2016)

Let $t_j := \mathrm{tr} A_j$, and $t := \sqrt{\frac{1}{m}\sum_j t_j^2}$. Then

$$\sum_j t_j A_j = \frac{mt^2}{d} I$$

and the operators

$$E_j = \frac{dt_j}{mt^2} A_j$$

constitute a POVM. If $t_j = const$ this POVM is also a conical 2-design.

Lemma (Index of coincidence)

The index of coincidence equals k_s, i.e.

$$\sum_j \mathrm{tr}^2[A_j \rho] = k_s$$

for every pure state ρ.

Some classes of conical 2-designs

- weighted projective 2-designs:

$$\sum_{k=1}^{n} w(k)\rho_k^{\otimes t} = \binom{d+t-1}{t}^{-1} P_s, \quad \text{where} \sum_k w(k) = 1$$

- homogeneous conical 2-designs ($\mathrm{tr}A_j = const$ and $\mathrm{tr}(A_j^2) = const$)
  - SIMs and MUMs (arbitrary rank generalization of SICs and MUBs)
    - anti-SICs and anti-MUBs: POVMs consisting of subnormalized projectors onto $d-1$-dimensional subspaces orthogonal to the states defining original POVM
    - depolarized SICs, MUBs and their anti- counterparts: convex mixtures of original POVM elements with identity

Lemma

For any conical 2 design $A_1, \ldots, A_m$ let a_j and b_j be the maximum and minimum eigenvalues of operator A_j, respectively. Then the affine combination $A_{j,\lambda} := \lambda A_j + (1-\lambda)\frac{1}{m}\mathbb{I}$ with

$$\lambda \in \left[\max_j \frac{1}{1-ma_j}, \min_j \frac{1}{1-mb_j}\right],$$

is also a conical 2-design.

In particular, for rank-1 A_j, $\frac{t_j\mathbb{I}-A_j}{d-1}$ (the anti- A_j) is a conical 2-design.

$V := \{\pi_i, \rho_i\}_{i=1}^{l}$ – an ensemble of initial states, $\Pi = \{\Pi_j\}_{j=1}^{k}$ – a POVM

How much information can be extracted from V by measurement Π?

Definition

The **mutual information between V and Π** is given by

$$I(V,\Pi) := \sum_{i=1}^{l} \eta\left(\sum_{j=1}^{k} P_{ij}\right) + \sum_{j=1}^{k} \eta\left(\sum_{i=1}^{l} P_{ij}\right) - \sum_{i=1}^{l}\sum_{j=1}^{k} \eta(P_{ij}),$$

where $P_{ij} = \pi_i \mathrm{tr}(\rho_i \Pi_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, and $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

What is the capability of extracting information by given measurement?

Definition

The **informational power** of Π is denoted by $W(\Pi)$ and given by

$$W(\Pi) := \max_{V-\mathrm{ensemble}} I(V,\Pi).$$

M. Dall'Arno, G.M. D'Ariano, and M.F. Sacchi, Phys. Rev. A **83**, 062304 (2011)

O. Oreshkov, J. Calsamiglia, R. Muñoz Tapia, and E. Bagan, New J. Phys. **13**, 073032 (2011)

- There exists a *maximally informative ensemble* consisting of pure states only
- The informational power and the *Shannon entropy of measurement* Π, defined for an initial state ρ by $H(\rho,\Pi) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho\Pi_j))$ are related by

$$W(\Pi) \le \ln k - \min_{\rho} H(\rho,\Pi)$$

with equality iff there exists an ensemble $V = \{\pi_i, \rho_i\}_{i=1}^{l}$ such that the states ρ_i are minimizers of $H(\cdot,\Pi)$ and $\mathrm{tr}((\sum_{i=1}^{l} \pi_i\rho_i)\Pi_j) = 1/k$ for every j

- The cases in which the informational power has been computed analytically so far:
  - all highly symmetric POVMs in dimension 2: seven sporadic measurements[3], including the 'tetrahedral' SIC-POVM[1,2], and one infinite series[2,3],
  - all SIC-POVMs in dimension three[4],
  - the POVM consisting of four MUBs in dimension three[5]
  - the Hoggar SIC-POVM in dimension eight[6]

[1] M. Dall'Arno, G.M. D'Ariano, and M.F. Sacchi, Phys. Rev. A **83**, 062304 (2011)
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[4] AS, J. Phys. A **47**, 445301 (2014)
[5] M. Dall'Arno, Phys. Rev. A **90**, 052311 (2014)
[6] W. Słomczyński and AS, arXiv:1512.01735

The optimization method based on the Hermite interpolation

- a sequence of points: $a \leq t_1 < t_2 < \ldots < t_m \leq b$
- a sequence of positive integers: $k_1, k_2, \ldots, k_m$
- a real valued function $f \in \mathcal{C}^D([a, b])$, where $D := k_1 + k_2 + \ldots + k_m$

There exists a unique polynomial p of degree less than D that agree with f at t_i up to a derivative of order $k_i - 1$ (for $1 \leq i \leq m$), that is,

$$p^{(k)}(t_i) = f^{(k)}(t_i), \qquad 0 \leq k < k_i. \tag{1}$$

We will need the following formula for the error in Hermite interpolation:

Lemma

For each $t \in (a, b)$ there exists $\xi \in (a, b)$ such that $\min\{t, t_1\} < \xi < \max\{t, t_m\}$ and

$$f(t) - p(t) = \frac{f^{(D)}(\xi)}{D!} \prod_{i=1}^{m} (t - t_i)^{k_i}. \tag{2}$$

Assume that all the derivatives of f of even order are strictly negative in (a, b) and these of odd order greater than 1 are strictly positive (holds true for η):

$$f^{2l}(x) < 0, \quad f^{2l+1}(x) > 0 \quad \text{for } x \in (a, b),\ l = 1, 2, \ldots . \tag{3}$$

Moreover, let us assume that

$$k_i := \begin{cases} 1, & \text{if } t_i \in \{a, b\} \\ 2, & \text{otherwise} \end{cases} . \tag{4}$$

Observation

Under above assumptions, the Hermite polynomial p interpolates f

1. *from below, if $t_1 = a$,*
2. *from above, if $t_1 > a$.*

Moreover, $f(t) = p(t)$ if and only if $t = t_i$ for some $i = 1, \ldots, m$.

Theorem (Informational power of depolarized anti-SICs)

Let A_j be a d-dimensional anti-SIC and $A_{j,\lambda} := [\lambda A_j + (1-\lambda)\mathbb{I}/d]$ be its depolarized counterpart. Then the informational power of $A_{j,\lambda}$ is given by

$$\ln d + \frac{1}{d}\eta\left(\frac{1-\lambda}{d}\right) + \frac{d^2-1}{d}\eta\left(\frac{d^2-1+\lambda}{d(d^2-1)}\right)$$

and the maximally informative ensemble consists of the corresponding SIC ensemble.

Sketch of the proof.

The optimal probability distribution consists of the smallest achievable and d^2-1 equal other probabilities. We apply the Hermite interpolation method. □

Theorem (Informational power of depolarized qubit SICs)

The informational power of any depolarized 2-dimensional SIC $A_{j,\lambda}$ is given by

$$\ln 2 + \frac{1}{2}\eta\left(\frac{1-\lambda}{2}\right) + \frac{3}{2}\eta\left(\frac{3+\lambda}{6}\right)$$

and the maximally informative ensemble consists of the corresponding anti-SIC ensemble.

Sketch of the proof.

The optimal probability distribution consists of the smallest achievable and 3 equal other probabilities. We apply the Hermite interpolation method. □

Theorem (Informational power of depolarized qubit complete MUBs)

The informational power of any depolarized 2-dimensional complete MUB $A_{j,\lambda}$ is given by

$$\ln(2) + \frac{1}{3}\eta\left(\frac{1-\lambda}{2}\right) + \frac{4}{3}\eta\left(\frac{1}{2}\right) + \frac{1}{3}\eta\left(\frac{1+\lambda}{2}\right)$$

and the maximally informative ensemble consists of the corresponding MUB ensemble.

Sketch of the proof.

The optimal probability distribution consists of the smallest achievable, tle largest achievable and 4 equal other probabilities. We apply the Hermite interpolation method. □

Theorem (Informational power of depolarized qutrit SICs)

The informational power of any depolarized 3-dimensional SIC $A_{j,\lambda}$ is given by

$$\ln(3) + \eta\left(\frac{1-\lambda}{3}\right) + 2\eta\left(\frac{2+\lambda}{6}\right)$$

and the maximally informative ensemble is an orthonormal basis consisting of elements orthogonal to exactly 3 elements of original SIC.

Sketch of the proof.

The optimal probability distribution consists of 3 the smallest achievable, and 6 equal other probabilities. We apply the Hermite interpolation method. □

Theorem (Informational power of depolarized qutrit complete MUBs)

The informational power of any depolarized 3*-dimensional complete MUB* $A_{j,\lambda}$ *is given by*

$$\ln(3) + \eta\left(\frac{1-\lambda}{3}\right) + 2\eta\left(\frac{2+\lambda}{6}\right)$$

and the maximally informative ensemble consists of the elements of corresponding Hesse SIC.

Sketch of the proof.

The optimal probability distribution consists of 4 the smallest achievable, and 8 equal other probabilities. We apply the Hermite interpolation method. □

Theorem (Informational power of depolarized Hoggar SICs)

The informational power of any depolarized Hoggar SIC $A_{j,\lambda}$ is given by

$$\ln 8 + \frac{7}{2}\eta\left(\frac{1-\lambda}{8}\right) + \frac{9}{2}\eta\left(\frac{9+7\lambda}{72}\right)$$

and the maximally informative ensemble consists of the elements of the dual Hoggar SIC.

Sketch of the proof.

The optimal probability distribution consists of 28 the smallest achievable, and 32 equal other probabilities. We apply the Hermite interpolation method. □