Informational power of the Hoggar SIC-POVM

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19.10.2015

Outline

Definition

A discrete quantum measurement is described by a **positive operator valued measure** (POVM) $\Pi = \{\Pi_j\}_{j=1}^{k}$, where Π_j are positive semidefinite operators on $\mathbb{C}^d$ that satisfy the identity decomposition

$$\sum_{j=1}^{k} \Pi_j = \mathbb{I}.$$

Definition

A **symmetric informationally complete POVM (SIC-POVM)** consists of d^2 subnormalized rank-one projections $\Pi_j = |\phi_j\rangle\langle\phi_j|/d$ with equal pairwise Hilbert-Schmidt inner products:

$$\mathrm{tr}(\Pi_i^* \Pi_j) = \frac{|\langle\phi_i|\phi_j\rangle|^2}{d^2} = \frac{1}{d^2(d+1)} \text{ for } i \neq j$$

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SIC-POVMs

- first studied in the context of *equiangular lines* in $\mathbb{C}^d$ or *complex spherical 2-designs* (e.g. Hoggar's papers 1978-82)
- extensively examined by Zauner in his PhD Thesis (1999) under the name of *regular quantum designs with degree 1*
- independently studied by Renes et al. (2003), the notion of *SIC-POVMs* introduced
- the existence of SIC-POVMs in every dimension – still an open problem
  - analytical solutions known for $d = 2 - 16, 19, 24, 28, 35, 48$
  - numerical confirmation up to $d = 67$
- $d = 3$ – the only known dimension with infinite family of nonequivalent SIC-POVMs
- all known SIC-POVMs are group-covariant
  - the Hoggar SIC-POVM ($d = 8$) is the only known SIC-POVM that is not group-covariant with respect to the finite Weyl-Heisenberg group
  - prime dimensions: group-covariance implies WH-covariance (Zhu, 2010)
  - there are exactly 3 *supersymmetric* SIC-POVMs (any two rays can be transformed into any two rays via symmetry group's action): $d = 2$, $d = 3$ (known as the Hesse configuration), $d = 8$ (the Hoggar lines); Zhu (2015)

The Hoggar SIC-POVM (the Hoggar lines)

- Constructed by Hoggar (1978) as complexification of diameters of certain quaternionic polytope in $\mathbb{H}^4$.
- Zauner in his PhD thesis indicated that this set can be obtained by taking the orbit of fiducial vector

$$\psi = \frac{1}{\sqrt{6}}(1+i, 0, -1, 1, -i, -1, 0, 0)^T$$

under action of the 3-fold tensor product of the 2×2 Weyl matrices.

- New simple construction by Jedwab and Wiebe (2015) using Hadamard matrices.

For which states of the system before the measurement, the uncertainty of the measurement outcomes is minimal?

We shall minimize the following quantity:

Definition

The **Shannon entropy of measurement** Π is defined by:

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho \Pi_j)),$$

for an initial state ρ, where $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

Example

For projective rank-1 measurement $\Pi := \{|\psi_j\rangle\langle\psi_j|\}_{j=1}^{d}$ the entropy is minimized (and equal 0) by the states defining PVM.

Definition

For an ensemble $V := \{\pi_i, \rho_i\}_{i=1}^{l}$ of initial states ρ_i with *a priori* probabilities π_i the **mutual information between V and Π** is given by

$$I(V,\Pi) := \sum_{i=1}^{l} \eta\left(\sum_{j=1}^{k} P_{ij}\right) + \sum_{j=1}^{k} \eta\left(\sum_{i=1}^{l} P_{ij}\right) - \sum_{i=1}^{l}\sum_{j=1}^{k} \eta(P_{ij}),$$

where $P_{ij} = \pi_i \mathrm{tr}(\rho_i \Pi_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k$.

The maximum of $I(V,\Pi)$ taken over all possible ensembles is called the **informational power** of Π and denoted by $W(\Pi)$.

- There exists a *maximally informative ensemble* consisting of pure states only
- If Π is group-covariant then

$$\max_{V-\text{ensemble}} I(V,\Pi) = \max_{\rho}(\ln k - H(\rho,\Pi)) = \ln k - \min_{\rho} H(\rho,\Pi)$$

Informational power of SIC-POVMs

- Upper bounded by $\ln \frac{2d}{d+1}$. The demanded probability distribution should be of the form:

$$(\frac{2}{d(d+1)}, \dots, \frac{2}{d(d+1)}, 0, \dots, 0)$$

with $d(d-1)/2$ zeros.
- The upper bound is satisfied for $d = 2, 3$.
- Some preliminary numerical calculations in dimensions 4 to 6 indicate that it is not always the case.

Construction by Jedwab and Wiebe

A **complex Hadamard matrix** $H = (h_{ij})_{i,j=1}^{d}$ is a $d \times d$ matrix such that $|h_{ij}|^2 = 1$ for $i, j = 1, \ldots, d$ and

$$HH^{\dagger} = d\mathrm{I}_d.$$

In particular, if all the entries are in $\{-1, 1\}$, then H is called **real Hadamard matrix** and then we get:

$$\sum_{j=1}^{d} h_{ij} h_{ij} = d \qquad (1)$$

and

$$\sum_{j=1}^{d} h_{ij} h_{kj} = 0, \quad \text{for } i \neq k. \qquad (2)$$

Two Hadamard matrices H and H' are called **equivalent** if there exist permutation matrices P, P' and diagonal unitary matrices D, D' such that $H' = DPHP'D'$.

Construction by Jedwab and Wiebe – continued

- Construct set $H(v) := \{H(v)_{jk}\}_{j,k=1}^{d}$ of d^2 vectors such that $(H(v))_{jk}$ is the j-th row of complex Hadamard matrix H with the k-th coordinate multiplied by $v \in \mathbb{C}$.

Theorem (Jedwab,Wiebe)

$H(v)$ is a set of d^2 equiangular lines (i.e. $H(v)$ defines a SIC-POVM) if and only if

- $d = 2$ *and* $v \in \{(1/2)(1 \pm \sqrt{3})(1+i), (-1/2)(1 \pm \sqrt{3})(1+i), (1/2)(1 \pm \sqrt{3})(1-i), (-1/2)(1 \pm \sqrt{3})(1-i)\}$, *or*
- $d = 3$ *and* $v \in \{0, -2, 1 \pm \sqrt{3}i\}$, *or*
- $d = 8$, *H is equivalent to a real Hadamard matrix and* $v \in \{-1 \pm 2i\}$.

In particular, the obtained set of equiangular lines for $d = 8$ is unitarily equivalent to the set of Hoggar lines.

Informational power of the Hoggar SIC-POVM

Theorem

Let H' be a complex Hadamard matrix in dimension $d \in \{2, 8\}$, equivalent to a real Hadamard matrix H, and v such that $H'(v) = \{H'(v)_{jk}\}_{j,k=1}^{d}$ is a set of equiangular vectors. Then the entropy of SIC-POVM generated by these vectors is minimized by states defined by $H'(\bar{v})$.

Proof.

Without loss of generality we can assume that $H' = H$.
Let us denote the canonical ONB in $\mathbb{C}^d$ by $(e_l)_{l=1}^{d}$. Then

$$(H(v))_{jk} = \sum_{l=1}^{d} h_{jl} e_l + (v-1) h_{jk} e_k .$$

We will show that for every $j, k = 1, \ldots, d$ the term $|(H(v))_{jk} \cdot (H(\bar{v}))_{mn}|^2$ can attain exactly two values: 0 and $const(d, v)$.

Proof – continued.

$$\begin{aligned} |(H(v))_{jk} \cdot (H(\bar{v}))_{mn}|^2 &= |\sum_{l,r=1}^{d} h_{jl} h_{mr} e_l \cdot e_r + (v-1)^2 h_{jk} h_{mn} e_k \cdot e_n \\ &+ \sum_{l=1}^{d} (v-1)(h_{jl} h_{mn} e_l \cdot e_n + h_{jk} h_{ml} e_k \cdot e_l)|^2 \\ &= |d\delta_{jm} + (v-1)(h_{jn} h_{mn} + h_{jk} h_{mk}) + (v-1)^2 h_{jk} h_{mn} \delta_{kn}|^2, \end{aligned}$$

where in the last equation we have used (1).
For $m \neq j$ and $n \neq k$ we get

$$|(H(v))_{jk} \cdot (H(\bar{v}))_{mn}|^2 = |(v-1)(h_{ik} h_{jk} + h_{il} h_{jl})|^2.$$

For given j and k and for all $m \neq j$ there exist exactly $d/2$ such n that the above sum is equal to 0 (see (2)). Otherwise it is $\pm 4|v-1|^2$.
For $m = j$ and $n \neq k$ we get

$$|(H(v))_{jk} \cdot (H(\bar{v}))_{jn}|^2 = |d + 2v - 2|^2.$$

Proof – continued.

For $m \neq j$ and $n = k$ we get

$$|(H(v))_{jk} \cdot (H(\bar{v}))_{mk}|^2 = |v^2 - 1|^2.$$

Finally, for $m = j$ and $n = k$ we get

$$|(H(v))_{jk} \cdot (H(\bar{v}))_{jk}|^2 = |d - 1 + v^2|^2.$$

Straightforward calculations show that for the values of v provided by Jedwab and Wiebe all the non-zero values of $|(H(v))_{jk} \cdot (H(\bar{v}))_{mn}|^2$ are the same and depend only on d and v. In consequence we obtain that this term attain two values: 0, with multiplicity $(d-1)d/2$, and $const(d, v)$, with multiplicity $d(d+1)/2$. Thus the upper bound is satisfied. □

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