Nuclear norm of tensors and entanglement

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Overview

- Norms
- Spectral and nuclear norms for matrices and tensors
- Entanglement measure for biparttie and *d*-partite state
- Banach's theorem for symmetric states
- Nonseparability measure for *d*-partite density matrices
- NP-hardness of tensor and nuclear norms
- Banach's theorem for symmetric *p*-partite density matrices
- Quantum channels decrease nonseparability measure

A primer on norms I

$\mathbb{F} = \mathbb{C}, \mathbb{R}$, - basic fields,

$\mathbb{F}^m$: $|x\rangle := \mathbf{x} = (x_1, \ldots, x_m)^\top$, $\langle x| := \mathbf{x}^* = (\bar{x}_1, \ldots, \bar{x}_m)$

$\nu : \mathbb{F}^n \to [0, \infty)$ a norm if

$\nu(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$, $\nu(\mathbf{x} + \mathbf{y}) \le \nu(\mathbf{x}) + \nu(\mathbf{y})$, $\nu(a\mathbf{x}) = |a|\nu(\mathbf{x})$

$B_\nu := \{\mathbf{x} \in \mathbb{F}^n, \nu(\mathbf{x}) \le 1\}$-unit ball,

$S_\nu := \{\mathbf{x} \in \mathbb{F}^n, \nu(\mathbf{x}) = 1\}$-unit sphere

Ext B_ν-extreme points of B_ν, Ext $B_\nu \subset S_\nu$

$\nu^\vee$-the dual norm:

$\nu^\vee(\mathbf{x}) = \max\{\Re(\mathbf{y}^*\mathbf{x}), \mathbf{y} \in B_\nu\} = \max\{|\mathbf{y}^*\mathbf{x}|, \mathbf{y} \in S_\nu\}$

$= \sup\{Re(\mathbf{y}^*\mathbf{x}), \mathbf{y} \in \mathrm{Ext}\ B_\nu\}$,

A primer on norms II

$(\nu^\vee)^\vee = \nu$

$S \subset \mathbb{F}^n$ is balanced: $aS = S \ \forall a \in \mathbb{F}, |a| = 1$

spanning: $\mathrm{span} S = \mathbb{F}^n$

If S is compact, balanced, spanning:

$\nu(\mathbf{x}) = \max\{\Re \mathbf{y}^* \mathbf{x}, \mathbf{y} \in S\} \ \forall \mathbf{x} \in \mathbb{F}^n$ is a norm

and $B_{\nu^\vee} = \mathrm{conv}\ S$

Eucledian norm $\|\mathbf{x}\| := \sqrt{\mathbf{x}^* \mathbf{x}}$ is self dual

$B(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{F}^n, \|\mathbf{y} - \mathbf{x}\| \le r\}, r \ge 0$

$\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$, $\|\mathbf{x}\|_2 = \|\mathbf{x}\|$

Spectral norm and SVD for matrices

$\mathbb{F}^{m\times n}$ - space of $m\times n$ matrices $A=[a_{ij}]_{i=j=1}^{m,n}$

$\langle A,B\rangle := \mathrm{Tr}(AB^*)$,

$\|A\|_F=\sqrt{\mathrm{Tr}\,AA^*}$ - Frobenius norm

$\|A\|=\sigma_1(A):=\max_{\|\mathbf{x}\|\le 1}\|A\mathbf{x}\|$ - spectral, or operator, or ℓ_2 norm of A

$\Omega_{m,n,\mathbb{F}}:=\{\mathbf{u}\mathbf{v}^*\in\mathbb{F}^{m\times n}, \|\mathbf{u}\|=\|\mathbf{v}\|=1\}$ balanced compact set

$\|A\|=\max\{\Re(\mathrm{Tr}(A\mathbf{v}\mathbf{u}^*))=\max\{\Re(\mathbf{u}^*A\mathbf{v}),\mathbf{u}\mathbf{v}^*\in\Omega_{m,n,\mathbb{F}}\}$

SVD decomposition:

$A=U\Sigma V^*, UU^*=I_m, VV^*=I_n, \Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_{\min(m,n)})$

$A=\sum_{i=1}^r\sigma_i\mathbf{u}_i\mathbf{v}_i^*, \sigma_1\ge\ldots\ge\sigma_r>0, r=\mathrm{rank}\,A$

Nuclear Norm for Matrices

Nuclear norm: $\|A\|_{nuc} := \sum_{i=1}^{\min(m,n)} \sigma_i(A)$

F-norm: $\|A\|_F^2 = \sum \sigma_i^2$

p-Schatten norm $\|A\|_{p,S} = (\sum_{i=1}^{n} \sigma_i(A)^p)^{\frac{1}{p}}$

If A is real valued then $\|A\|, \|A\|_{nuc}$ over real same as over complex

Complexity of computation of $\|A\|, \|A\|_{nuc}$ is $O(mn), O(\min(m,n)mn)$

Importance of matrix nuclear norm in missing entry completion:

Netflex problem

Minimal characterization of matrix nuclear norm

$B_{nuc} := \{A \in \mathbb{F}^{m\times n} : \ \|A\|_{nuc} = \sum_{i=1}^r \sigma_i(A) \le 1\}$

$A = \|A\|_{nuc} \sum_{i=1}^r \frac{\sigma_i}{\|A\|_{nuc}} \mathbf{u}_i \mathbf{v}_i^*$

The set of extreme points of B_{nuc} is $\Omega_{m,n,\mathbb{F}}$

Characterization of spectral norm gives $\|\cdot\|^\vee = \|\cdot\|_{nuc}$

$\|A\|_{nuc} = \min\{\sum_{i=1}^N \|\mathbf{x}_i\|\|\mathbf{y}_i^*\|, \sum_{i=1}^N \mathbf{x}_i\mathbf{y}_i^* = A\}$

Proof $\|A\|_{nuc} = \|\sum_{i=1}^N \mathbf{x}_i\mathbf{y}_i^*\| \le \sum_{i=1}^N \|\mathbf{x}_i\mathbf{y}_i^*\|_{nuc} = \sum_{i=1}^N \|\mathbf{x}_i\|\|\mathbf{y}_i^*\|$

Caratheodory: $\dim \mathbb{R}^{m\times n} = mn \Rightarrow$ it is sufficient $N = mn + 1$

$\dim_{\mathbb{R}} \mathbb{C}^{m\times n} = 2mn \Rightarrow$ it is sufficient $N = 2mn + 1$

Alternating Minimization Method for computing $\|A\|_{nuc}$

Choose $\mathbf{y}_1, \ldots, \mathbf{y}_N \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ in general position (at random)

$L(A, \mathbf{y}_1, \ldots, \mathbf{y}_N) := \{X := [\mathbf{x}_1 \ldots \mathbf{x}_N] \in \mathbb{F}^{m\times N}, A = \sum_{i=1}^N \mathbf{x}_i \mathbf{y}_i^*\}$

Find $\min_{X \in L(A, \mathbf{y}_1, \ldots, \mathbf{y}_N)} \|X\|_y, = [\mathbf{x}_{1,1} \ldots \mathbf{x}_{N,1}], \|X\|_y := \sum_{i=1}^n \|\mathbf{x}_i\| \|y_i\|$

Now repeat this minimization with respect to $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and so on

Entanglement measure of bipartite states

$A = [a_{ij}] \in \mathbb{C}^{m\times n}$ is a bipartite state:

$1 = \|A\|_F = \|A\|_F^2 = \sum_{i=1}^{\min(m,n)} \sigma_i(A)^2$

Entanglement:

$\eta(A) := \|A\|_{nuc} - \|A\|_F = \|A\|_{nuc} - 1 = \sum_{i=1}^{n} \sigma_i(A) - \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i(A)^2}$

$0 \le \eta(A) \le \sqrt{\min(m,n)}$

$\eta(A) = 0$ - A unentangled, product state

$\eta(A) = \sqrt{\min(m,n)}$ - maximally entangled state:

Example: $I_m \oplus 0_{m\times(n-m)}$ maximally entangled

Tensor notations

Indices: $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, $[m] := \{1, \ldots, m\}$

$J = \{j_1, \ldots, j_k\} \subset [d]$

Tensors: $\otimes_{i=1}^d \mathbb{F}^{m_i} = \mathbb{F}^{m_1 \times \ldots \times m_d} = \mathbb{F}^{\mathbf{m}}$

Contraction of $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{m}}$ with $\mathcal{X} = [x_{i_{j_1},\ldots,i_{j_k}}] \in \otimes_{j_p \in J} \mathbb{F}^{m_{j_p}}$:

$\mathcal{T} \times \mathcal{X} = \sum_{i_{j_p} \in [m_{j_p}], j_p \in J} t_{i_1,\ldots,i_d} x_{i_{j_1},\ldots,i_{j_k}} \in \otimes_{l \in [d] \setminus J} \mathbb{F}^{m_l}$

$\|\mathcal{T}\| = \sqrt{\mathcal{T} \times \bar{\mathcal{T}}}$ - Hilbert-Schmidt norm of $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$

$\langle \mathcal{T}, \mathcal{S} \rangle := \mathcal{T} \times \bar{\mathcal{S}}$ inner product in $\mathbb{C}^{\mathbf{m}}$

Tensor nuclear and spectral norms - 3-tensor

$\mathcal{A} \in \mathbb{F}^{l\times m\times n}, \qquad \Omega_{m,n,l,\mathbb{F}} := \{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} \in \mathbb{F}^{m\times n\times l}, \|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\| = 1\}$

$\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = |x\rangle|y\rangle|z\rangle$

$\|\mathcal{A}\|_{\sigma,\mathbb{F}} := \max_{\mathbf{x},\mathbf{y},\mathbf{z}\neq 0} \frac{\Re\langle\mathcal{A},\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\|} =$

$\max\{|\langle\mathcal{A},\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\rangle|, \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} \in \Omega_{m,n,l,\mathbb{F}}\}$ -spectral norm

$\|\mathcal{A}\|_{nuc,\mathbb{F}} := \min\left\{\sum_{i=1}^r \|\mathbf{x}_i\|\|\mathbf{y}_i\|\|\mathbf{z}_i\| : \mathcal{A} = \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i,\ r\in\mathbb{N}\right\}$

$\|\cdot\|_{nuc,\mathbb{F}} = \|\cdot\|_{\sigma,\mathbb{F}}^{\vee}$,

$\mathrm{Ext}\ B_{\|\cdot\|_{nuc,\mathbb{F}}} = \Omega_{m,n,l,\mathbb{F}}$.

Hillar-Lim: Spectral norm is NP-hard to compute

Friedland-Lim: Nuclear norm is NP-hard to compute

Measure of entaglement of tensor

$\mathcal{A} \in \otimes_{j=1}^{d} \mathbb{C}^{n_j}$ d-partite state: $1 = \|\mathcal{A}\|_{HS} = \langle \mathcal{A}, \mathcal{A} \rangle$

Physical interpretation of $\|\mathcal{A}\|_{nuc}$: the minimal ℓ_1 energy to express

d-partite state as linear combination of pure states

Measure of entanglement:

$\eta(\mathcal{A}) := \|\mathcal{A}\|_{nuc} - 1 \geq 0$

$\eta(\mathcal{A}) = 0$ iff $\mathcal{A}$ product space

What is the maximum entanglement and the maximal entangled state?

Symmetric states and Banach's theorem

$\mathcal{A} = [a_{i_1,\ldots,i_d}] \in \otimes^d \mathbb{C}^n$ symmetric:

$a_{i_{\sigma(1)},\ldots,i_{\sigma(d)}} = a_{i_1,\ldots,i_d}$ for each permutation σ

A symmetric state is a linear combination of pure symmetric states:

$\otimes^d \mathbf{x} = |x\rangle^d$

Banach 1938: $\|\mathcal{A}\|_\sigma = \max\{|\langle \mathcal{A}, \otimes^d \mathbf{x}\rangle|, \|\mathbf{x}\| = 1\}$

Cor. : The minimal decomposition of symmetric state $\mathcal{A}$

for nuclear norm is achieved by pure symmetric states

Computational advantage for computing $\eta(\mathcal{A})$.

4-tensors and bi-partite density matrices

$\mathbb{C}^{m\times m} \supset \mathbb{H}^{m\times m} \supset \mathbb{H}_+^{m\times m} \supset \mathbb{H}_{+,1}^{m\times m}$

Hermitian, positive definite and density matrices

$A = [a_{ij}] \in \mathbb{F}^{m\times n}, B = [b_{kl}] \in \mathbb{F}^{p\times q}$, Kronecker product $A \otimes B$

$$\begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} = [c_{(i,k)(j,l)}] \in \mathbb{F}^{(mp)\times(nq)}, c_{(i,k)(j,l)} = a_{ij}b_{kl}$$

Viewing $\mathcal{C} := [c_{(i,k)(j,l)}] \in \mathbb{F}^{m\times p\times n\times q}$ we get $\mathbb{F}^{m\times n} \otimes \mathbb{F}^{p\times q} \sim \mathbb{F}^{m\times p\times n\times q}$

$\mathcal{C} = [c_{i,k,j,l}] \in \mathbb{C}^{m\times n\times m\times n}$ is called:

Bi-hermitian: $c_{i,k,j,l} = \bar{c}_{j,l,i,k}$ for all i, j, k, l - $C = [c_{(i,k)(j,l)}]$ hermitian

Positive definite: C is hermitian positive semi-definite,

bi-partite density matrix: $\mathrm{Tr}\, C = 1$

Bi-partite separable states and nuclear norm

Separable states in $\mathbb{C}^{m\times n\times m\times n}$:

$\mathrm{S}(m,n) := \mathrm{conv}\,\{(\mathbf{x}\otimes\mathbf{x}^*)\otimes(\mathbf{y}\otimes\mathbf{y}^*), \|\mathbf{x}\| = \|\mathbf{y}\| = 1\} \subset \mathbb{H}_{mn,+,1}$

For $\mathcal{A} = [a_{ikjl}] \in \mathbb{C}^{m\times n\times m\times n}$ define $\mathrm{tr}(\mathcal{A}) := \sum_{i,j} a_{ijij}$ $(= \mathrm{Tr}\, C)$

Note $\mathrm{Tr} \otimes_{j=1}^4 \mathbf{x}_j = (\mathbf{x}_3^\top \mathbf{x}_1)(\mathbf{x}_4^\top \mathbf{x}_1) \le \prod_{i=1}^4 \|\mathbf{x}_i\|$

THM: $|\,\mathrm{Tr}\,\mathcal{A}| \le \|\mathcal{A}\|_{nuc}$ equality iff $\mathcal{A} = t\mathcal{B}$, $\mathcal{B} \in \mathrm{S}(m,n)$

Cor. A bipartite density matrix is separable iff its nuclear norm is 1

Measurement of separability: $\eta(\mathcal{A}) := \|\mathcal{A}\|_{nuc} - 1$.

In the minimum decomposition of nonseparable density matrices as

product hermitian states appear negative definite terms

NP-hardness results

Gurvits 2003: Weak membership in $S(m, n)$ is NP-hard $\Rightarrow$:

Membership in the unit ball of nuclear norm on $\mathbb{C}^{m\times n\times m\times n}$ NP-hard

Friedland-Lim: Weak membership is NP-hard

Outline of NP-hardness arguments:

A weak membership in B_ν is polynomial iff

A weak memebership in $B_{\nu^\vee}$ is polynomial

$\nu(\mathbf{x})$ can be approximated polynomially

computation of spectral norm of bipartite density matrix is NP-hard

WMEM for nuclear tensor norm is NP-hard

Finding an ϵ approximation to spectral norm is NP-hard

Equivalent to WMEM in the unit ball of nuclear norm

Finding an WMEM in the uni spectral norm is NP-hard

WMEM for nuclear norm is NP-hard

WMEM of nuclear norm is equivalent to approximation of nuclear norm

Approximation of nuclear norm is NP-hard

Banach's theorem for symmetric d-partite density matrices

$\mathcal{A} = [a_{i,j,k,l}] \in \mathbb{C}^{m\times n\times m\times n}$ bipartite density matrix

Variational characterization of spectral norm as maximal λ:

$\mathcal{A} \times \mathbf{y}^* \otimes \mathbf{x} \otimes \mathbf{y} = \lambda \mathbf{x},\ \mathcal{A} \times \mathbf{x}^* \otimes \mathbf{x} \otimes \mathbf{y} = \lambda \mathbf{y},\ \|\mathbf{x}\| = \|\mathbf{y}\| = 1$

$\mathcal{A}$ symmetric if $m = n$ and $a_{i,j,k,l} = a_{j,i,k,l}$ for all indices

Banach's theorem 1: $\|\mathcal{A}\|_\sigma = \max\{\langle \mathcal{A}, \bar{\mathbf{x}} \otimes \bar{\mathbf{x}} \otimes \mathbf{x} \otimes \mathbf{x}\rangle, \|\mathbf{x}\| = 1\}$

Maximal eigenvector: $\mathcal{A} \times \bar{\mathbf{x}} \otimes \mathbf{x} \otimes \mathbf{x} = \|\mathcal{A}\|_\sigma \mathbf{x}$

Banach's theorem 2: minimal decomposition of symmetric p-density

matrix for nuclear norm given by pure symmetric states $\otimes^p \bar{\mathbf{x}} \otimes^p \mathbf{x}$

Quantum channels decrease inseparability measure

$Q : \mathbb{C}^{M\times M} \to \mathbb{C}^{N\times N}$ completely positive:

$Q(\rho) = \sum_{i=1}^{k} A_i \rho A_i^*, \; A_i \in \mathbb{C}^{N\times M}$

Q-quantum channel: $\sum_{i=1}^{k} A_i^* A_i = I_M$ (trace preserving)

Q-unital channel: $\sum_{i=1}^{k} A_i A_i^* = I_N$

Bipartite Q: $A_i = A_{i,1} \otimes A_{i,2}$ for all i

Thm 1: Biparite quantum channel decreases nuclear norm of $\mathcal{A}$

Bipartite quantum channel decreases nonseparability $\eta(\mathcal{A})$

Thm 2: Biparite unital channel decreases spectral norm of $\mathcal{A}$

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