Generalized entropies: its connections with Shannon and Kolmogorov-Sinai entropies and an invariant based on this concept

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seminarium „Chaos i informacja kwantowa”,
Kraków, 4.11.2013

Let (X, Σ, μ) be a probability space
$T : X \to X$ - (measurable) measure-preserving transformation
For a finite partition $\mathcal{P} = \{E_1, ..., E_k\}$ we consider a join partition

$$\mathcal{P}_n := \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P} := \{\bigcap_{i=0}^{n-1} A_i, \text{ where } A_i \in T^{-i}\mathcal{P} \text{ for } i = 0, ..., n-1\}$$

where

$$T^{-i}\mathcal{P} = \{T^{-i}E_1, ..., T^{-i}E_k\}.$$

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Dynamical and Kolmogorov-Sinai entropy

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$$H(\mathcal{P}_n) = \sum_{A \in \mathcal{P}_n} \eta(\mu(A)).$$

We define the entropy of the transformation T with respect to the partition $\mathcal{P}$ (the dynamical entropy) as

$$h_\mu(T, \mathcal{P}) = h(\mathcal{P}) = \limsup_{n \to \infty} \frac{1}{n} H(\mathcal{P}_n). \quad (1)$$

For a given system (X, Σ, μ, T) we define the Kolmogorov-Sinai entropy of T (with respect to μ) as

$$h_\mu(g, T) = \sup_{\mathcal{P}-\text{finite}} h_\mu(T, \mathcal{P}). \quad (2)$$

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Reasons of the generalization

- KS entropy is an isomorphism invariant; what about systems with equal entropy (e.g. zero entropy systems)
- how important are properties of $\eta(x) = -x \ln x$ for the entropy in dynamical systems
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$$\mathcal{G}_0 = \{\mathrm{g}\colon [0,1] \mapsto \mathbb{R},\ \mathrm{g}\ -\text{concave},\ \mathrm{g}(0) = \lim_{\mathrm{x}\to 0^+} \mathrm{g}(\mathrm{x}) = 0\}.$$

Let $\mathrm{g} \in \mathcal{G}_0$. We define the g-entropy of the transformation T with respect to the partition $\mathcal{P}$ (the dynamical g-entropy) as

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$$\mathcal{G}_0^0 = \{g \in \mathcal{G}_0 \mid C(g) = 0\},$$

e.g. $g(x) = \frac{x - x^\alpha}{\alpha - 1}, \alpha > 1$, $g(x) = x\ln(1 - \ln x)$;

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Theorem

Let $g \in \mathcal{G}_0^\infty$ and T be an aperiodic, surjective automorphism of a Lebesgue space (X, Σ, μ) and let $\gamma \in \mathbb{R}$. Then there exists a partition $\mathcal{P} = \{E, X \setminus E\}$, such that

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Measure-theoretic g-entropy

Following the Kolmogorov proposition we take the supremum over all partitions of dynamical g-entropy of a partition. For a given system $(\mathrm{X}, \Sigma, \mu, \mathrm{T})$ we define

$$\mathrm{h}_\mu(\mathrm{g}, \mathrm{T}) = \sup_{\mathcal{P}-\text{finite}} \mathrm{h}_\mu(\mathrm{g}, \mathrm{T}, \mathcal{P}) \tag{4}$$

and call it the measure-theoretic g-entropy of transformation T with respect to measure μ.

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Main theorem

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Fact

Let $g \in \mathcal{G}_0^\infty$. If (X, T) is aperiodic and surjective, then $h_\mu(g, T) = \infty$.

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Rates of g-entropy convergence

F. Blume “Possible rates of entropy convergence” Ergod. Th.& Dynam. Sys. 17. 45–70 (1997)

Let (X, T) be a measure-preserving system, T –bijective, $(a_n)_{n\in\mathbb{N}}$ a monotone increasing sequence with $\lim_{n\to\infty} a_n = \infty$ and $c \in (0, \infty)$. Let P be a class of partitions of X. Let $g \in \mathcal{G}_0$. We say that (X, T) is of type $(LS(g) \geq c)$ for $((a_n), P)$ if

$$\limsup_{n\to\infty} \frac{H(g, \mathcal{P}_n)}{a_n} \geq c \quad \text{for all } \mathcal{P} \in P$$

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Rates of g-entropy convergence

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Choice of P and (a_n)

- class of partitions

  $$P(X) := \{\mathcal{P} | \mathcal{P} = \{E, X \backslash E\} \text{ for some } E \in \Sigma \text{ with } 0 < \mu(E) < 1\}.$$

  If (X, T) and (Y, S) are isomorphic measure-preserving systems, then (X, T) is of type $(LS(g) \geq c)$ for $((a_n), P(X))$ iff (Y, S) is of type $(LS(g) \geq c)$ for $((a_n), P(Y))$.
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  if (X, T) has zero entropy and $g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{Sh}$, we have

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Negative result

Theorem

Let $g \in \mathcal{G}_0$ with $C(g) > 0$. If (X, T) is an aperiodic measure-preserving system and (a_n) is a positive monotone increasing sequence with $\lim_{n \to \infty} \frac{a_n}{n} = 0$, then (X, T) is not of type $(LI(g) < \infty)$ for $((a_n), P(X))$.

Can we get something new?

Table: Connections between η-entropy types and g-entropy types of convergence

	$g \in \mathcal{G}_0^0$	$g \in \mathcal{G}_0^{Sh}$	$g \in \mathcal{G}_0^\infty$
$LS(\eta) \leq c$	$LS(g) = 0$	$LS(g) \leq C(g) \cdot c$	$LS(g) \leq \infty$, $LS(g) = \infty$
$LS(\eta) \geq c$	$LS(g) = 0$, $LS(g) \geq 0$	$LS(g) \geq C(g) \cdot c$	$LS(g) = \infty$
$LS(\eta) < \infty$	$LS(g) = 0$	$LS(g) < \infty$	$LS(g) \leq \infty$, $LS(g) = \infty$
$LS(\eta) = \infty$	$LS(g) < \infty$, $LS(g) \leq \infty$, $LS(g) = \infty$	$LS(g) = \infty$	$LS(g) = \infty$

$g_0 \in \mathcal{G}_0^0$ – aperiodic systems

- Every subshift over two symbols is of type $(\mathrm{LS}(g) \le 1)$ for $\left(\left(\varphi(2^{-n})\right)_{n=1}^{\infty}, P(X)\right)$.
- Let $g_0(x) = x \log_2(1 - \log_2 x)$.

Theorem

If (X, T) is aperiodic and measure-preserving and $\phi\colon [1, \infty) \mapsto (0, \infty)$ is an increasing function with $\int_1^{\infty} \frac{\phi(x)}{x^2} dx < \infty$, then for every $\mathcal{P}$ such that $\lim_{n\to\infty} \max\{\mu(A) | A \in \mathcal{P}_n\} = 0$, we have

$$\limsup_{n\to\infty} \frac{H(g_0, \mathcal{P}_n)}{\phi(n g_0(1/n))} = \infty.$$

If $\int_1^{\infty} \frac{\phi(x)}{x^2} dx = \infty$, then there exists a weakly mixing system (X, T) and a meas. set E such that $0 < \mu(E) < 1$ and $\lim_{n\to\infty} H(g_0, \mathcal{P}_n)/\phi(n g_0(1/n)) = 0$.

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$g_0 \in \mathcal{G}_0^0$ – completely ergodic systems

- If $([0,1],T)$ is completely ergodic, then there exists such a sequence (a_n) with $\lim_{n\to\infty} \frac{a_n}{n} = 0$, $\lim_{n\to\infty} a_n = \infty$, that for every $\mathcal{P} \in P([0,1])$ we have

$$\liminf_{n\to\infty} \frac{H(g_0, \mathcal{P}_n)}{a_n} \geq 1.$$

- Under the assumption of the previous theorem there exists (a_n) such that (X,T) is of type $LS(\eta) = \infty$ for $((a_n), P([0,1]))$.
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- We may construct a class of rank-one weakly mixing systems where we can use type $(\mathrm{LI(g)} \geq \mathrm{c})$ for $((\mathrm{a_n}), \mathrm{P(X)})$ to distinguish systems. Depending on the choice of g we may use other than η-entropy types of convergence to differ rank-one systems.

Additional assumptions on g

$g'(0) = \infty$ ($g \in \mathcal{G}_0^0$)

Subderivativity of g

The crucial property of the static g-entropy is the following:

$$H(g, \mathcal{P} \vee \mathcal{Q}) \leq H(g, \mathcal{P}) + H(g, \mathcal{Q}|\mathcal{P})$$

It is sufficient that for every $x, y \in [0, 1]$ function g fulfills the following condition

$$g(xy) \leq xg(y) + yg(x), \tag{5}$$

The condition is not easy to check. On the other hand if we want to construct such a function we can define

$$g(x) := xh(-\ln x),$$

where $h : (0, \infty) \mapsto \mathbb{R}$ is a concave, subadditive and increasing with $\lim_{x\to\infty} h(x) = \infty$, $\lim_{x\to\infty} \frac{h(x)}{x} = 0$.

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Examples of subderivative functions

- for $h(x) = \ln(1+x)$, we get $g(x) = x\ln(1-\ln x)$,
- for $h(x) = x^{\alpha}$, $\alpha \in (0,1)$ we have $g(x) = x(-\ln x)^{\alpha}$,
- if

$$h(x) := \begin{cases} x, & \text{for } x \in [0,1) \\ 2^{-k}x + 2^{k+1} - 2, & \text{for } x \in [4^k, 4^{k+1}), \quad k = 0, 1, \dots \end{cases}$$

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