Quantifying Quantum Entanglement

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Quantum Chaos

* **Quantum** analogues of **classically chaotic** systems can be described by **generic** (random) matrices

* A given initial state $|\phi_0\rangle$ is evolved by a unitary evolution operator U into a **generic** pure state $|\psi\rangle = U|\psi_0\rangle$.

Quantum Information

* Several problems of **quantum information** can be solved using notions of **random** states and **random** operations.

Example: **Non-additivity of capacity of quantum channels**,
Hastings 2009 - a non-constructive proof obtained using random channels in large dimensions...

Quantum Chaos and Quantum Information: Mathematical aspects

Quantum (and Classical) Chaos

* Dynamical systems, ergodic theory, periodic orbits and kneading theory
* Geometric Quantisation and theory of Hilbert spaces
* Quantum ergodicity & number theory
* Coherent state theory
* Universality and **random matrix theory**

Quantum Information

* Mathematical theory of information processing
* Operator theory and positive maps
* Geometry of convex sets in large dimensional spaces
* Quantum Entanglement $\Rightarrow$
  group theory and invariants with respect to group actions
* Entanglement measures $\Rightarrow$ theory of foliations

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $N = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := \; -\mathrm{Tr}\, \sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...

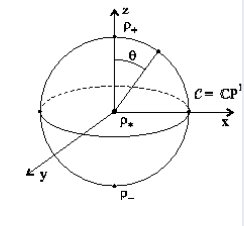
Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Space of normalized complex pure states for an arbitrary N:

Since $\langle\psi|\psi\rangle = 1$ a **normalized** state belongs to the **sphere** S^{2N-1}.

Two states equal up to a phase are identified, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$, so the set of states is equivalent to the **complex projective space** $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

$N = 2$: For **qubit** = quantum bit the word **geometry** can be treated literally!



$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

$\mathbb{C}P^1$ = **Bloch sphere of $N = 2$ pure states**

Unitary evolution

Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}(|\psi\rangle, |\varphi\rangle) \;:=\; \arccos |\langle\psi|\varphi\rangle|$$

Unitary evolution = an isometry

Let $U = \exp(-iHt)$. Then $|\psi'\rangle = U\,|\psi\rangle$.
Since $|\langle\psi|\varphi\rangle| = |\langle\psi|U^\dagger U|\varphi\rangle|$ any unitary evolution is an isometry
(with respect to any standard distance !)

'Quantum Chaology'

a) How an **isometry** may lead to classically chaotic dynamics?
The limits $t \to \infty$ and $N \to \infty$ do not commute.
b) Properties of an evolution operator U corresponding to the classically **chaotic system** can be described by a **random unitary matrix** distributed according to the **Haar measure on** $U(N)$**.**

Random Pure states in $\mathcal{H}_N$

'Quantum chaotic' dynamics **(pseudo-random evolution)**

described by a **random unitary** matrix U acting on a pure state produces (almost surely) a **'generic pure state'** $|\psi\rangle = U|\phi_0\rangle$.

• Formally one defines an (unique) **Fubini–Study measure** μ on complex projective spaces which is **unitarily invariant**: for any (measurable) set A of states one requires $\mu(A) = \mu(U(A))$.

• This measure covers the entire space $\mathbb{C}P^{N-1}$ **uniformly**, and for $N = 2$ it is just equivalent to the uniform, **Lebesgue measure on the sphere** S^2.

How to obtain numerically a **random pure state** $|\psi\rangle$ **?**

a) Take a column (a row) of a **random unitary** U so that $|\psi\rangle = U|i\rangle$.
b) generate N **independent complex random numbers** z_i according to the **normal** distribution. Write $|\psi\rangle = \sum_{i=1}^{N} c_i|i\rangle$ where the expansion coefficients read $c_i = z_i/\sqrt{\sum_i |z_i|^2}$.

Pure States

Bi-partite systems: $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition $|\psi\rangle = A_{ij}|i\rangle \otimes |j\rangle = \sum_{m=1}^{N} \sqrt{\lambda_m}\, |m\rangle \otimes |m\rangle$

Entanglement measures:

functions of the **Schmidt vector** $\bar{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$

Entanglement entropy $E(\phi) = S(\bar{\lambda})$ (**Shannon** entropy)

generalized - " - $E_q(\phi) = S_q(\bar{\lambda})$ (**Renyi**/Tsallis entropies)

concurrence $C = \sqrt{2(1 - \sum_m \lambda_m^2)}$ (function of E_2 (or purity)

Symmetric functions, e.g. $\mu_2 = \sum_m \sum_n \lambda_m \lambda_n$.

Multipartite systems

Geometric entanglement measure (**Wei, Goldbart, 2003)** -

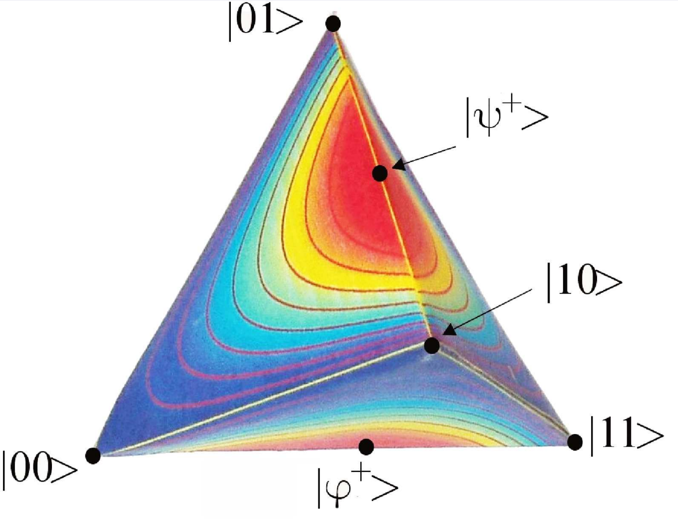
$G(\psi) = 1 - \max_{\chi_{\rm sep}} |\langle\psi|\chi_{\rm sep}\rangle|^2$

a 'distance' to the closest **separable** state)

$|\chi_{\rm sep}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \otimes \cdots \otimes |\phi_K\rangle$

Entanglement of two real qubits

Entanglement entropy at the thetrahedron of $N = 4$ real pure states



Wawel Castle, Cracow, Poland

Mixed states

Definition of Entanglement

- **separable mixed states:** $\rho_{\text{sep}} = \sum_j p_j \, \rho_j^A \otimes \rho_j^B$
- **entangled mixed states:** all states not of the above product form.

Entanglement measures for mixed states:

generalization of a measure M from pure states to mixed states:

$$M(\rho) := \min_{\mathcal{E}} \sum_i p_i M(|\psi_i\rangle) \qquad \textbf{(convex roof)}$$

where **ensemble** $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Examples:

Entropy of formation, $E(\rho) \ = ;\min_{\mathcal{E}} \sum_i p_i E(|\psi_i\rangle)$

Concurrence of formation, $C(\rho) \ = \ \min_{\mathcal{E}} \sum_i p_i C(|\psi_i\rangle)$

Measurable entanglement measures

measuring purity $\mathrm{Tr}\rho^2$ – two copies in a coincidence experiment
higher moments: $\mathrm{Tr}\rho^k$ – k copies in a coincidence experiment

P. Horodecki, A. Ekert, 2002

Measurable bounds for concurrence

purity difference (**Mintert, Buchleitner, 2007**)
$C(\rho) \geq \mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_A^2$ and $C(\rho) \geq \mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_B^2$
where $\rho_A = \mathrm{Tr_B}\rho$ and $\rho_B = \mathrm{Tr_A}\rho$ are partial traces...

Generalization for other maps by **Augusiak, Lewenstein 2009**

Entanglement witness

State ρ is entangled if there exists an observable W (**entanglement witness**) such that for any separable σ one has $\mathrm{Tr}W\sigma \geq 0$ and

$$w := \mathrm{Tr}W\rho < 0$$

In general, **the smaller** w (more negative), the **larger entanglement**

(but this approach depends on the choice of W)

Main Motivation

1. Need for a sensitive entanglement test (measure) for multipartite systems,
2. which will work for dimensions larger than $N = 2$,
3. and will be experimentally accessible for pure and for mixed states.

Optical effects at the Market Square in **Cracow**, Poland

A collective entanglement measure

Maximal collectibility for a K–partite pure state

Let $|\Psi\rangle \in \mathcal{H} = \mathcal{H}^A \otimes H^B \ldots \otimes \mathcal{H}^K$,

where all dimensions are equal, $\dim\left(\mathcal{H}^J\right) = N$.

Select N separable pure states, $|\chi_j^{sep}\rangle = |a_j^A\rangle \otimes \ldots \otimes |a_j^K\rangle$, where $|a_j^J\rangle \in \mathcal{H}^J$ with $j = 1, \ldots, N$ and $J = A, \ldots, K$.

The states are **mutually orthogonal**, $\langle a_j^J | a_k^J \rangle = \delta_{jk}$ for $J = A, \ldots, K$.

Define the **maximal collectibility**

$$Y^{\max}[|\Psi\rangle] \; := \; \max_{|\chi^{sep}\rangle} \prod_{j=1}^{N} |\langle \Psi | \chi_j^{sep} \rangle|^2 .$$

In "Geometric entanglement measure" $G(\Psi)$

we look for the largest projection onto a **single** product state'

so the overlap $|\langle \Psi | \chi^{sep} \rangle|^2$ is **smallest** for a highly **entangled** state.

In **collectibility** $Y(\Psi)$ the **product** of N overlaps,

$|\langle \Psi | \chi_1^{sep} \rangle|^2 \cdots |\langle \Psi | \chi_N^{sep} \rangle|^2$ is **largest** for a highly **entangled** state $|\Psi\rangle$.

Upper bound for collectibility

For any pure state $|\Psi\rangle$ we show the following bound

$$Y^{\max}[|\Psi\rangle] \ \leq \ N^{-N}.$$

Setting $Z^{\max} = -\ln Y^{\max}$ this relation takes the from

$$Z^{\max}[\Psi] \ \geq \ N \ln N$$

analogous to the **entropic uncertainty relation**.

It is saturated for the maximally entangled state, $|\Psi_+\rangle = \frac{1}{\sqrt{N}} \sum_i |i, i\rangle$ (bi-partite case) and a generalized GHZ state

$$|GHZ\rangle_K = \frac{1}{\sqrt{N}} \sum_i |i\rangle_A \otimes \cdots \otimes |i\rangle_K \quad \text{ in } K\text{–partite case.}$$

Collectibility for separable states

For any separable state $|\Psi_{\text{sep}}\rangle$ the following bound holds

$$Y^{max}[|\Psi_{\text{sep}}\rangle] \ \leq \ N^{-N \cdot K} \quad \Longrightarrow \quad \text{A } \textbf{separability} \text{ criterion:}$$

If $Y^{max}(|\Psi\rangle) \geq N^{-NK}$ then the state $|\Psi\rangle$ is **entangled**.

A partial collectibility for a general $N^{\otimes K}$ system

To find $Y^{\max}[|\Psi\rangle]$ we need to optimize over a base consisting of N separable states $|\chi_j^{sep}\rangle$

Start with a **single** optimization over the subspace $\mathcal{H}^A$, and define the **partial collectibility**, $Y_a[|\Psi\rangle] := \max_{|a^A\rangle} \prod_{j=1}^N |\langle\Psi|\chi_j^{sep}\rangle|^2$ parameterized by the set a of N product states $|a_j^B\rangle \otimes \ldots \otimes |a_j^K\rangle$, with $j = 1, \ldots, N$.

By construction one has: $\max_a Y_a[|\Psi\rangle] = Y^{max}[|\Psi\rangle]$.

Collectibility for two qubit system, $K = 2$ and $N = 2$

$$Y_a\left[|\Psi\rangle\right] = \frac{1}{4}\left(\sqrt{G_{11}G_{22}} + \sqrt{G_{11}G_{22} - |G_{12}|^2}\right)^2,$$

where $G_{jk} = \langle\varphi_j|\varphi_k\rangle$ is a **Gram matrix** among projected states,
so that $|\varphi_j\rangle = \langle a_j^B|\Psi_{AB}\rangle \in \mathcal{H}^A$.

Collectibility for two qubit system, $K = 2$ and $N = 2$

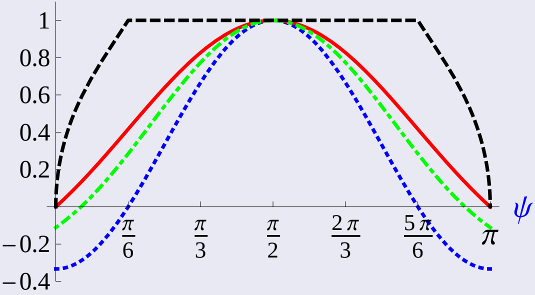
Write a bi-partite pure state in its **Schmidt form**

$$|\Psi_{AB}\rangle = (U_A \otimes U_B)\left[\cos\left(\frac{\psi}{2}\right)|00\rangle + \sin\left(\frac{\psi}{2}\right)|11\rangle\right]$$

Direct optimization gives its **collectibility**

$$Y^{max}(\psi) = [1 + \sin(\psi)]^2/16.$$

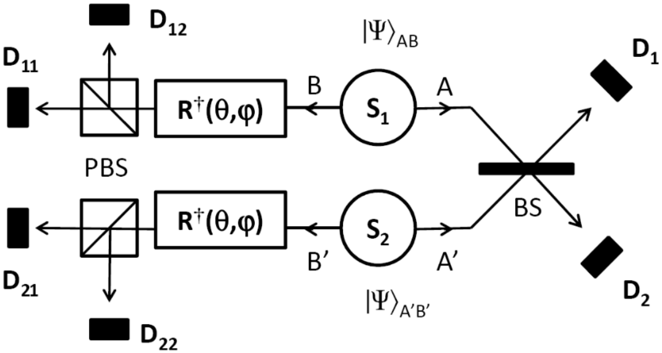
We get also $Y^{min}(\psi) = \sin^2(\psi)/4$ and the mean value $\bar{Y}(\psi)$ averaged over random position of the detector bases $|a_1^B\rangle$ and $|a_2^B\rangle$.



Maximal (red), **average** (green) and **minimal** (blue) values of the rescaled (partial) **collectibility**, $[16Y(\psi) - 1]/3$. Positive values identify **entanglement**.

Experimental photonic setup: measurement of $|G_{12}|^2$

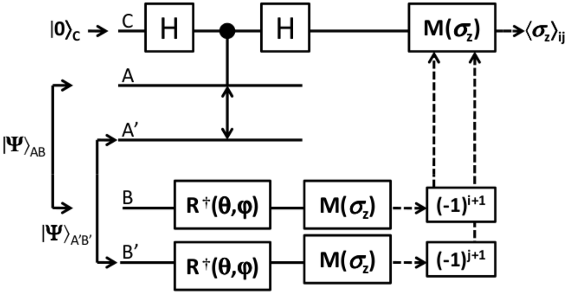
On the **left side** B the statistics of pairs of clicks after projections onto detectors are measured,
on the **right side** A the **Hong–Ou–Mandel interference** is performed.



The number $|G_{12}|^2$ is equal to the probability of the pair of the **clicks at** B multiplied by that of a **double click at** A.

Quantum network to measure $|G_{12}|^2$

exploiting two copies of an analyzed state $|\Psi\rangle_{AB}$, a control qubit $|c\rangle$ initially in state $|0\rangle$, controlled SWAP gate and two Hadamard gates.



Mean value of Pauli σ_z matrix of the **controlled qubit** $|c\rangle$ is measured under the condition that the chosen pair (i, j) of results is obtained in measurement of the same observables performed on both qubits B.

Purity assumption for $|\Psi\rangle_{AB}$ may be dropped at a price of performing two variants of the experiment in two complementary settings...

Collectibility for three qubit system, $K = 3$ and $N = 2$

The **collectibility** is maximal for a

a) GHZ state, $|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$,

and then $Y^{\max}[|GHZ\rangle] = 16/64{=}1/4$,

while for

b) W–state, $|W\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

it reads $Y^{\max}[|GHZ\rangle] = 9/64$. For a

c) bi–separable state, $|BS\rangle = |\Psi\rangle_{AB} \otimes |\phi\rangle_C$

one has $Y^{\max}[|BS\rangle] = 4/64 = 1/16$,

while for

d) separable state the **collectibility** reads, $Y^{\max}[|\Psi_{\rm sep}\rangle] = 1/64$.

Collectibility as a detector of the **genuine entanglement**

Thus any measured value of $Y^{\max}[|\Psi\rangle]$ above $1/16$
provides an evidence for **genuine three–party entanglement**
for the analyzed state $|\Psi\rangle$!

Market Square, **Cracow**, Poland

Concluding remarks

1. We introduced **collectibility** $Y^{\max}(\Psi)$ as a function of any pure state $|\Psi\rangle$ of a composed $N^{\otimes K}$ system,
2. **Collectibility** satisfies inequalities analogous to **entropic uncertainty relations**
3. The **partial collectibility** $Y_a(\Psi)$ labeled by parameters describing positions of detectors $|a_j^B\rangle, \ldots, |a_j^K\rangle$ is experimentally accessible for any K–qubit system,
4. An experimental photonic scheme based on **Hong–Ou–Mandel interferometry** is proposed to measure **collectibility** in a two-qubit system.
5. Results for **collectibility**, presented here for **pure states** only, can also be generalized for **non-ideal pure states** with **purity** less then one, $\mathrm{Tr}\rho^2 > 1 - \epsilon$.
6. Sufficiently large value of $Y_a(\Psi)$ detects **entanglement** in a bi-partite system and **genuine three–party entanglement** for an analyzed state $|\Psi\rangle$ of a three–partite system.

Cracow with the **Wawel Castle**
and the **Tatra mountains** in the background.