Spontaneous emission of nondispersive Rydberg wave packets

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Nondispersive electronic Rydberg wave packets may be created in atoms illuminated by a microwave field of circular polarization. We discuss the spontaneous emission from such states, and show that the elastic incoherent component (occurring at the frequency of the driving field) dominates the spectrum in the semiclassical limit, contrary to earlier predictions. We calculate the frequencies of single-photon emissions and the associated rates in the “harmonic approximation,” i.e., when the wave packet has approximately a Gaussian shape. The results agree well with exact quantum-mechanical calculations, which validates the analytical approach. [S1050-2947(98)04307-8]

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I. INTRODUCTION

Recent theoretical studies predicted the existence of nondispersive wave packets in atoms driven by a microwave radiation of either circular [1] or linear polarization [2]. Contrary to standard electronic wave packets which disperse in time when the binding force is nonlinear, these objects do not spread due to a nonlinear coupling between the atom and the driving field which locks the electronic motion onto the driving frequency. The phenomenon responsible for this locking is a 1:1 resonance between the internal (Kepler) and external (microwave) frequencies, creating a resonance island, at the center of which lies a stable periodic orbit. In the dressed-atom picture [3], a single eigenstate of the full “dressed” Hamiltonian represents the wave packet [2,4], immediately explaining the absence of spreading.

The very same microwave field is also responsible for the decay of the wave packet via ionization. As shown elsewhere [5], the typical ionization time of wave packets built from circular Rydberg states around a principal quantum number $n_\text{cl} = 60$ exceeds $10^5$ Kepler periods, often being longer than $10^6$ periods. The ionization occurs via a “chaos-assisted tunneling” mechanism [6] (the tunneling occurring from the stable island into the chaotic sea surrounding it). In effect, the lifetime against ionization is very sensitive to small changes of the parameters (microwave frequency $\Omega$, or its amplitude $F$) exhibiting large-scale fluctuations [6].

Another possible decay mechanism of the wave packet is of course the spontaneous emission of photons. For a circularly polarized microwave, the situation considered in this paper, it was discussed in Ref. [7]. Here we readdress this issue since, in our opinion, the results reported in Ref. [7] are incomplete. To this end, we use a different approach, borrowing heavily from the well-known treatment of resonance fluorescence of two-level atoms driven by a laser field [3] using the dressed-atom approach. The main advantage of the present approach is its generality: it can be used for any polarization of the microwave field, for any superposition of driving fields with various frequencies, and for any structure of electromagnetic modes surrounding the atom (as shown, e.g., in Ref. [8], the presence of a cavity can strongly affect the spontaneous emission of an atom). Also, it uses the common knowledge widely developed in the quantum optics community, and is thus accessible to a large audience. Let us also mention that spontaneous emission of nondispersive wave packets in a linearly polarized microwave was discussed in Ref. [9].

The paper is organized as follows. In Sec. II, we discuss the basic properties of the nondispersive wave packets (not considering spontaneous emission) from different points of view: the laboratory frame or rotating frame, the time-dependent or Floquet pictures, and the dressed-atom picture. The discussion is necessary for a proper understanding of spontaneous emission, which is done in Sec. III. Our results are compared to the earlier published results in Sec. IV. Section V is devoted to the validity of the harmonic approximation, by comparing our analytical results to “exact” numerical calculations. We conclude in Sec. VI.

II. NONDISPERSE WAVE PACKETS FROM DIFFERENT POINTS OF VIEW

We consider a hydrogen atom in the presence of a monochromatic driving microwave field of circular polarization, that is, an electromagnetic field which can be described by a single mode. In this section, we will not consider any coupling to other (initially empty) modes of the electromagnetic field. That will be discussed in Sec. III. We first treat the microwave field classically. A full quantum treatment of the microwave field is possible (see Sec. II.D), but not really needed as, for the usual microwave fields, the number of photons in the mode is enormous, and a classical treatment of the microwave field is possible.

A. Hamiltonian in the laboratory frame

With the previous assumption, the quantum Hamiltonian of the system is (in atomic units, neglecting relativistic and QED effects, assuming an infinite mass of the nucleus, and in the dipole approximation)

\begin{equation}
H_{cl} = \frac{p^2}{2} - \frac{1}{r} + F(x \cos \Omega t + y \sin \Omega t),
\end{equation}

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where $\Omega$ is the frequency of the microwave field circularly polarized in the $x$-$y$ plane, and $F$ its amplitude. The $x$ axis is chosen to coincide with the microwave electric field at $t = 0$.

### B. Floquet approach

Hamiltonian (1) is periodic in time with period $2\pi/\Omega$. Hence a Floquet approach [10] may be used: any solution of the time-dependent Schrödinger equation can be expanded as a linear combination of time-periodic states — called Floquet states of the system — with coefficients oscillating in time as $\exp(-iE_i t)$, where $E_i$ is called the quasienergy of the Floquet state (there is no explicit $h$ in the equations, as we are using atomic units where $\hbar = 1$). Obviously, the quasienergy is defined only modulo $\Omega$, making the Floquet spectrum periodic in energy with period precisely $\Omega$. The Floquet spectrum as well as the Floquet states can be obtained from the diagonalization of a suitable operator, namely, the Floquet Hamiltonian

$$\mathcal{H} = H_{cl} - i \frac{\partial}{\partial t}, \quad (2)$$

with time-periodic boundary conditions. A suitable basis for the Floquet Hamiltonian is the product of any basis in usual space — for example, the usual hydrogenic eigenbasis with bound states $|nlm\rangle$ — by the basis of time-periodic oscillating exponentials $\exp(iK\Omega t)$, with $K$ an integer. We denote $|nlm; K\rangle$ as such a basis. The next step is to expand the time-periodic Hamiltonian in a Fourier series, and calculate the nonzero matrix elements of the Floquet Hamiltonian. Because no higher harmonics of the microwave frequency exist in the Hamiltonian, only matrix elements changing $K$ by zero or 1 unit do not vanish. Moreover, in the specific case of circular polarization, the part in the Hamiltonian which increases (decreases) $K$ by one is proportional to $x - iy$ ($x + iy$), hence decreases (increases) the angular momentum of the electron along the $z$ axis by one unit. In other words, the Floquet Hamiltonian obeys the selection rule $\Delta(K + m) = 0$. This merely signifies that absorption of a $\sigma^\pm$-polarized microwave photon must increase the atomic angular momentum $L_z = xp_y - yp_x$ by one unit. The Floquet Hamiltonian thus has a block-diagonal structure, and the Hilbert space can be split into manifolds characterized by a constant value of $K = m + K$. In each block, the Floquet Hamiltonian is no longer periodic in energy. On the other hand, the Floquet Hamiltonian in blocks $K$ and $K'$ differs only by a constant energy term $(K' - K)\Omega$, which restores the total periodicity of the Floquet spectrum. However, diagonalization in a given block is sufficient to obtain all information on the system.

### C. Hamiltonian in the rotating frame

For a circularly polarized radiation, another possibility exists which has been frequently used in the past [1, 4–7, 11–21]. By passing to the noninertial frame rotating with frequency $\Omega$ using the unitary transformation $U = \exp(i\Omega t)$, one may remove the oscillatory time dependence from Eq. (1), obtaining, in the rotating frame, the Hamiltonian

$$H_{rot} = U H_{cl} U^\dagger + i U \frac{\partial U^\dagger}{\partial t} = \frac{p^2}{2} - \frac{1}{r} + Fx - \Omega L_z. \quad (3)$$

This time-independent Hamiltonian has some energy levels and corresponding eigenstates. Obviously, its spectrum is not $\Omega$ periodic. However, the link with the Floquet approach in the laboratory frame is easily established: there is a one-to-one correspondence between the eigenstates in the rotating frame and the Floquet states in the laboratory frame from the block $K = 0$. Indeed, consider an eigenstate (i.e., time-independent state) $|\phi_i\rangle$ of $H_{rot}$ with energy $E_i$. From Eq. (3), we deduce, straightforwardly,

$$\left(H_{cl} - i \frac{\partial}{\partial t}\right)|\phi_i\rangle = E_i|\phi_i\rangle, \quad (4)$$

which implies [see Eq. (2)] that $U^\dagger |\phi_i\rangle$ is a Floquet eigenstate with quasienergy $E_i$. From the definition $U^\dagger = \exp(-i\Omega L_z t)$, it is clear that the $K$th Fourier component of $U^\dagger |\phi_i\rangle$ is nothing but the component of $|\phi_i\rangle$ on the subspace with $L_z = m = -K$. In other words, $U^\dagger |\phi_i\rangle$ is a Floquet eigenstate in the laboratory frame lying in the block $K = m + K = 0$.

Of course, Floquet states belonging to the other blocks $K \neq 0$ can also be found from the time-independent Hamiltonian in the rotating frame. A first method is to use, instead of the operator $U$, a slightly different unitary equivalent operator $\exp(-iK\Omega t)$ (with $K$ an integer; it has the same time periodicity as $U$ itself) which leads to the following Hamiltonian in the rotating frame:

$$H_{rot} = \frac{p^2}{2} - \frac{1}{r} + Fx - \Omega (L_z - k). \quad (5)$$

Taking all possible positive and negative integer $k$ values, one recovers the full Floquet spectrum.

A second equivalent method is to keep the standard Hamiltonian in the rotating frame [Eq. (3)], and to consider states of the form $\exp(iK\Omega t)|\phi_i\rangle$. These are time-periodic Floquet (if $k$ is an integer), eigenstates of the Floquet Hamiltonian $\mathcal{H}$ [Eq. (2)] with energy $E_i + K\Omega$, belonging to the block $K = m + K = k$.

At a deeper level, this can be seen as a consequence of the unitary equivalence (which implies the same spectrum) for the Floquet Hamiltonians in the laboratory frame and in the rotating frame. The spectrum of the latter is here very simple: because $H_{rot}$ is time independent, it is simply the energy spectrum of $H_{rot}$ shifted by an arbitrary integer multiple of $\Omega$. For a more complicated case like — for example — a microwave with polarization close to circular, the Hamiltonian in the rotating frame is weakly time dependent, and its corresponding Floquet Hamiltonian (with the time-dependent term treated perturbatively) may be the simplest approach.

### D. Dressed-atom approach

An alternative view of this problem is to treat the microwave field quantum mechanically as being in either a coherent state or a pure Fock state with a large number of photons in the circularly polarized mode [22]. This approach is quite
common in quantum optics, and is known as the dressed-atom picture. Then the atom is not considered as an isolated quantum system, but rather a strongly coupled atom plus field system which is treated using quantum mechanics. The Hamiltonian of the atom plus field system is

\[ H_q = \frac{\mathbf{p}^2}{2} - \frac{1}{r} + V(d^-a^\dagger + d^+a) + \Omega a^\dagger a, \]

where \( a^\dagger \) and \( a \) are the creation and annihilation operators in the occupied, \( \sigma^+ \) circularly polarized mode, \( d_\pm = (x \pm iy)/\sqrt{2} \) are the corresponding components of the dipole moment operator, and \( V \) is the coupling strength, whose exact expression is given in Sec. III A but not important at the present level.

The Hamiltonian \( H_q \) is time independent and may be expanded in the basis \( \{ |nlm\rangle \otimes |N\rangle = |nlm;N\rangle \} \), where \( |nlm\rangle \) are hydrogenic states, and \( N \) denotes the number of microwave photons in the occupied mode. The diagonalization of the Hamiltonian yields then the dressed states; as its stands, the energy spectrum is not \( \Omega \) periodic.

The absorption of a circularly polarized microwave photon increases by 1 the projection \( L_z \) of the electron angular momentum on the \( z \) axis. In the quantum field basis this manifests itself in the fact that the Hamiltonian matrix is block diagonal. The Hilbert space splits into manifolds characterized by the total atom plus field projection of the angular momentum \( N \), i.e., basis states coupled within a given block have fixed \( N = m + N' \). \( N' \) is the total angular momentum projection on the \( z \) axis carried by both the electron and the photons.

When the number of photons is very large (\( N \gg 1 \)), we can neglect the variations in the nondiagonal matrix elements — proportional to either \( \sqrt{N} \) or \( \sqrt{N} + 1 \) — across neighboring \( N \) blocks. Then \( \sqrt{2}N/\sqrt{N} \) can be identified as the amplitude of the microwave field \( F \). With this approximation, the energy spectrum is (locally) \( \Omega \) periodic. As a matter of fact, the matrix elements of \( H_q \) are obviously equal to the ones of the classical Floquet Hamiltonian [Eq. (2)], already discussed in Sec. II B, provided that the quantum number \( K \) labeling the Fourier components in the Floquet picture is identified with the number of photons \( N \). This establishes — in the limit of a large number of photons — the one-to-one correspondence between the quasienergy spectrum of the Floquet Hamiltonian and the energy spectrum of the dressed atom, and also the correspondence between the Floquet eigenstates and the dressed-atom eigenstates. The two points of view are completely equivalent: in the following, we will use the dressed-atom picture, as it seems more convenient for incorporating the spontaneous emission of photons.

As a final remark, let us note that the previous discussion is already well known in a different context in quantum optics: for the simple case of a two-level atom coupled resonantly by an oscillatory field, and in the rotating-wave approximation. There a field-quantized or Floquet approach yields an energy spectrum periodic in \( \Omega \) — a sequence of doublets separated by the Rabi frequency \( \Omega \) — while semiclassically in the “rotating frame” [23] one obtains a single doublet of states.

E. Numerical calculation of the energy spectrum

As discussed in the three preceding subsections, there exists a one-to-one correspondence between the eigenstates in the rotating frame, the Floquet eigenstates, and the dressed states in the quantum language. Therefore, in the following we shall call the eigenenergies of Hamiltonians (2), (3), or (6) quasienergies, while referring to the eigenstates as Floquet eigenstates or dressed states.

As we are interested in the regime of highly excited Rydberg states (with a typical principal quantum number 60) driven by a resonant microwave field, the number of atomic states significantly coupled by the microwave field is enormous. For computational efficiency, it is better to use a Sturmian basis [4–6] rather than the usual atomic basis. The advantage of this is that it provides a discretization of the atomic continuum. Combined with the complex rotation technique, it allows us to find not only the energies of the dressed states but also their lifetimes against ionization by the microwave field [5,6]. To find several converged dressed states numerically, diagonalizations of very large matrices (dimension of the order of \( 10^6 \)) are necessary. As discussed above, there exists a full equivalence between diagonalization of the Floquet Hamiltonian, i.e., a clever use of the time-periodic structure of the problem and a diagonalization of the Hamiltonian in the rotating frame. The computational efficiencies of the two points of view are strictly equal. Our computer code [4–6] diagonalizes, in fact, \( H_{\text{rot}} \), contrary to the statement in Ref. [19].

F. Nondispersive wave packets

Some time ago, Klar [24] noticed that the Hamiltonian (3) in the rotating frame allows for the existence of a stable fixed (equilibrium) point in a certain range of microwave amplitude \( F \). The contribution of Ref. [1] was to notice that wave packets initially localized in the vicinity of this fixed point will not disperse (being bound by the fact that the fixed point is stable) for a very long time. In the laboratory frame, these wave packets, also called “Trojan states” in Ref. [1], appear as wave packets moving around the nucleus along the circular trajectory [the periodic orbit of Eq. (1)]. The existence of “stationary” wave packets — keeping their shape exactly during propagation along the circular orbit and hence not dispersing at arbitrarily long times (neglecting the very slow ionization) — was later found in Ref. [4]. These are simply some among the eigenstates of the dressed atom; by construction, they are strictly periodic functions of time.

Semiclassically, the wave packets are manifestations of the motion inside the stable primary 1:1 resonance island when the microwave frequency matches the electronic frequency. Similar states have been studied earlier for model one-dimensional driven systems [25]. While for other models, the wave packet itself resembles every period of the driving perturbation (while changing shape within one period [2,25]), the wave packets in a circular polarization keep the same shape at all times. This is due to the conservation of \( N \).

Quantum mechanically, any dressed eigenstate may be viewed as a linear combination of atomic states, combined with appropriate photonic states (in such a way that in the combination \( N \) is conserved). The wave-packet states are linear combinations of atomic Rydberg states coupled by
resonant microwaves. Thus the largest contribution to this combination comes from circular states with principal quantum number \( n \) close to
\[
n_0 = \Omega^{-1/3}. \quad (7)
\]
Note that \( n_0 \) is not necessarily an integer. For the particular case of the "ground-state wave packet" (with maximum localization near the fixed point), it is a superposition of mainly circular atomic states \( |n,n-1,n-1\rangle \) with a small contamination of other quasi-two-dimensional states of the form \( |n,1,1\rangle \), and even smaller contributions from states \( l \neq m \) extending significantly out of the \( xy \) plane.

The fixed point in the rotating frame is located at \( y_e = z_e = 0 \) and \( x_e \), satisfying
\[
1/x_e^2 + F - x_e \Omega^2 = 0. \quad (8)
\]

It is convenient to introduce [1] a dimensionless parameter
\[
q = \Omega^{-2} x_e^{-3}. \quad (9)
\]
Then \( q = 1 \) for \( F = 0 \), and the fixed point is stable for \( q \in [8/9, 1] \). It is also useful to define scaled variables which make the discussion of the results less dependent on the particular microwave frequency value [26]. We have already defined the effective principal quantum number \( n_0 \) [Eq. (7)].

The scaled microwave amplitude is defined as
\[
F_0 = F n_0^4 = F \Omega^{-4/3}. \quad (10)
\]

\( F_0 \) is a dimensionless parameter simply related to \( q \) [see Eq. (9)] via
\[
F_0 = (1 - q) q^{-1/3}. \quad (11)
\]

A possible way to create a nondispersive wave packet consists of the preparation of a circular state with a given principal quantum number \( n_0 \) followed by a smooth turn on of the microwave with frequency \( \Omega = 1/n_0^3 \) [4]. Such an approach has been tested numerically by integration of the time-dependent Schrödinger equation [27]. It has been shown that the wave packet may be obtained with about 90% efficiency.

During the switch-on of the microwave, two regimes may be identified. For \( q \) only slightly less than unity (small microwave amplitude), only the resonant coupling between circular states is important. The effective approximate Hamiltonian in the vicinity of the fixed point then corresponds to the Hamiltonian of a pendulum [12]. This is the standard description of the resonance at first order [28]. For low \( F_0 \), the resonance island is small in action variable and, in effect, the pendulum states must be quite extended in the conjugate direction. In the rotating frame, this manifests itself as the fixed point being marginally stable along the \( y \) direction at vanishing \( F_0 \). For too small an \( F_0 \), the binding force is too small, and the wave packet eigenstate can be hardly considered to be localized in the very vicinity of the fixed point and, in turn, cannot be considered as a localized wave packet in the laboratory frame.

With increasing microwave amplitude, the resonance island grows, and may support a few (depending on the size of the effective \( \hbar = 1/n_0 \) states). Then, for a sufficiently large \( F_0 \) (but not large enough to destabilize the resonance), the low excitations in the resonance island may be well described in the harmonic approximation. It is this regime of \( F_0 \) values which is of interest. The harmonic expansion of the Hamiltonian \( H_{\text{rot}} \) in the rotating frame around the equilibrium point allows one to find normal modes of the problem, and to obtain analytic predictions concerning the energies of wave packet eigenstates [1,4].

**G. Harmonic approximation**

The Hamiltonian \( H_{\text{rot}} \) can be expanded in powers of position and momentum operators in the vicinity of the fixed point. The first-order terms cancel because this is a fixed point. In the harmonic approximation, only second-order terms are kept (including of course crossed position-momentum terms), and we obtain
\[
H_{\text{rot}} = H_e + \frac{p^2}{2} + \Omega (x\bar{p}_y - \bar{y} p_x) + \frac{\Omega^2 q_y^2}{2} - \Omega^2 q_x^2 + \frac{\Omega^2 q_z^2}{2}, \quad (12)
\]

where
\[
E_e = \frac{1 - 4q}{2q^{2/3}} \Omega^{2/3} \quad (13)
\]
is the energy of the equilibrium point, and \((\bar{x}, \bar{y}, \bar{z}) = (x - x_e, y - y_e, z - z_e)\) denotes the displacement with respect to the fixed point.

The "diagonalization" of this Hamiltonian leads to the definition of the normal modes for the electronic motion. The calculation is tedious but straightforward: the frequencies of the three normal modes labeled by the \((+, -, z)\) indices are
\[
\omega_+ = \Omega \sqrt{2 - q \pm Q}, \quad (14)
\]
\[
\omega_- = \Omega \sqrt{q}, \quad (15)
\]
with a shorthand notation
\[
Q = \sqrt{9q^2 - 8q}. \quad (16)
\]
The corresponding creation and annihilation operators in the three normal modes are linear combinations of position and momentum operators. In the harmonic approximation, the motion perpendicular to the plane of microwave polarization decouples from the motion in the plane; hence the annihilation operator in the \( z \) mode has the simple expression
\[
b_z = \frac{q^{1/4} \sqrt{\Omega} z + iq^{-1/4} p_z / \sqrt{\Omega}}{\sqrt{2}}. \quad (17)
\]
The creation operator \( b_z^\dagger \), being the adjoint of \( b_z \), has a similar expression with an imaginary part of opposite sign. The \( \pm \) modes completely entangle the various coordinates and momenta \((\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y)\). They are neither the usual "car-
tesian” normal modes along $\tilde{x}$ and $\tilde{y}$, nor “circular” normal modes along $\tilde{x} \pm i\tilde{y}$, but intermediate combinations. The annihilation operators are

$$b_\pm = N_\pm \left( \sqrt{\Omega x - i \frac{(Q \mp 3q \pm 2)\Omega^{3/2}}{2\omega_\pm}} \tilde{y} - i \frac{(Q \pm q)\sqrt{\Omega_\pm}}{2q\omega_\pm} p_x + \frac{3q \mp Q_\pm}{2q\sqrt{\Omega_\pm}} \right),$$

with the normalization constants given by

$$N_\pm = \left( \frac{q^2(2Q^2 - q + 2 \pm 3(1 - 2q)Q)}{8Q^2(1 - q)} \right)^{1/4}.$$  (19)

The Hamiltonian in the harmonic approximation can be expressed in terms of creation and annihilation operators of the electronic normal modes $[1,4.5]$

$$H_\text{h} = \sum_{s} (\omega_s b_s^\dagger b_s + 1/2) - \omega_a (b_a^\dagger b_a + 1/2) + \omega_s (b_s^\dagger b_s + 1/2) + \omega_a (b_a^\dagger b_a + 1/2).$$  (20)

Note the unusual minus sign in front of the $\omega_a$ term. This comes from the fact that the stable fixed point is not a minimum of the Hamiltonian. However, the presence of crossed momentum-position terms assures the complete stability of the equilibrium point versus any small perturbation, in complete analogy with a usual potential minimum.

Let us denote the atomic eigenstates in the harmonic approximation as $|n_+, n_-, n_z; N\rangle$ where $n_i$ denotes the excitation in a given $\pm, z$ mode $[4.5]$. We will also denote the Floquet or dressed-atom eigenstates (thus including the microwave field degree of freedom) as $|n_+, n_-, n_z; N\rangle$. The corresponding energies are

$$E(n_+, n_-, n_z; N) = E_a + N\Omega + (n_+ + \frac{1}{2})\omega_+ - (n_- + \frac{1}{2})\omega_- - (n_z + \frac{1}{2})\omega_.  \quad (21)$$

It has been verified that exact Floquet quasienergies, corresponding to dressed states localized in the vicinity of the equilibrium point in the rotating frame, agree with the harmonic prediction [Eq. (21)] to within 10% of the mean level spacing $[4.5]$ for wave packets corresponding to $n_0 \approx 60$ or higher. This indicates that the harmonic approximation is quite satisfactory for sufficiently large $n_0$ (sufficiently small $\Omega$). In particular, the “ground state” $|0,0,0\rangle$ corresponds to the most tightly bound state, centered at the fixed point in the rotating frame. Its wave function has a Gaussian shape in the harmonic approximation. The corresponding exact Floquet (dressed) state $[5]$ is a nondispersive wave packet, since in the laboratory frame it moves in time around the nucleus with frequency $\Omega$ following the stable circular periodic orbit. It is precisely the spontaneous emission from this state which will be discussed in the following sections.

The partial scheme of levels resulting from $H_\text{b}$ is shown in Fig. 1 for a few values of the “total” angular momentum $N$. The neighboring ladders of states are shifted with respect to each other by the microwave frequency $\Omega$ corresponding to different decoupled blocks of the Floquet Hamiltonian.

FIG. 1. Schematic energy levels of a hydrogen atom dressed by a microwave field of circular polarization, with a Hamiltonian given by Eq. (6). The total angular momentum $N$ (atom plus field) is a constant of motion, so that the spectrum can be split into separate ladders. The consecutive ladders have the same structure, simply separated (in energy) by the microwave frequency $\Omega$. Here we have represented three consecutive ladders. If all ladders are taken into account, the spectrum is $\Omega$ periodic, and coincides with the Floquet spectrum of the classical Hamiltonian [Eq. (2)]. Alternatively, the spectrum in each ladder can be thought of as the energy spectrum of the Hamiltonian in a rotating frame [Eq. (3)]. Spontaneous emission of photons (represented by arrows in the figure) from a state in ladder $N$ is only possible to another state lying at a lower energy in either ladder $N-1$ (with $\sigma^+$ polarization), ladder $N$ (with $\pi$ polarization), or ladder $N+1$ (with $\sigma^-$ polarization). Hence $\sigma^+$ transitions are favored. In the specific case of the ground nondispersive wave packet $|0,0,0\rangle$ in the harmonic approximation, only three transitions, represented by the solid arrows, are allowed at frequencies $\Omega - \omega_+, \Omega$, and $\Omega + \omega_-$, all of them being $\sigma^+$ polarized.

Transitions between quasienergy levels (Floquet or dressed states) may be introduced by some additional perturbation. In particular, one may consider the so-called Floquet spectroscopy $[29]$, absorption or stimulated emission between dressed states introduced by an additional laser or microwave field.

It seems quite natural to study the possible effects of spontaneous emission of these states within the very same harmonic approximation. This allows one to obtain close analytic expressions for the spontaneous decay rate. The validity of the harmonic approximation has been challenged strongly by Farrell and co-workers $[16-20]$. By comparison of spontaneous spectra obtained in the harmonic approximation with full quantum-mechanical results, we show later that their claims are not justified.

III. SPONTANEOUS EMISSION OF THE WAVE PACKET IN THE DRESSED-ATOM PICTURE

A. General properties

To treat spontaneous emission, one has to consider the atom interacting not only with a single mode [treated either classically in Eq. (1) or quantum mechanically in Eq. (6)] but the full electromagnetic field. The corresponding Hamiltonian reads, in atomic units $[30]$,

$$H = H_f + r \cdot E(0) + H_f$$

(22)
where subscript \( i \) stands for \( \text{cl} \) or \( \text{q} \) corresponding to the classical or quantum version of the description of the occupied microwave mode, respectively. \( \mathbf{r} \) is the usual dipole operator, \( \mathbf{E}^{(\mathbf{r})} \) denotes the electric-field operator at position, \( \mathbf{r} \) and \( H_J \) is the free-field Hamiltonian. Both \( H_J \) and \( \mathbf{E}^{(\mathbf{r})} \) can be expanded over modes of the field. While in Ref. [7], the expansion in spherical waves is used (which facilitates explicit transformation to the rotating frame), we expand the field into standard plane waves [3]. We normalize the modes per unit in \( k \) space (related to the frequency in atomic units by \( \omega = \alpha k \), where \( \alpha = e^2/\hbar c = 1/137.036 \) is the fine-structure constant), then

\[
\mathbf{E}^{(\mathbf{r})} = \sum_j \int \frac{\sqrt{\omega}}{2\pi} [a_j(\mathbf{k}) \mathbf{e}_j \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{r}) + \text{H.c.}] d^3k, \tag{23}
\]

where H.c. stands for hermitian conjugation, \( a_j(\mathbf{k}) \) is the annihilation operator in the mode characterized by wavevector \( \mathbf{k} \) and polarization \( \mathbf{e}_j \) (the sum over \( j \) is over the two possible orthogonal polarizations). Note that in Eq. (22) the electric-field operator is evaluated at origin, consistently with the dipole approximation assumed in this paper. Similarly the free-field Hamiltonian is

\[
H_J = \sum_j \int \omega a_j(\mathbf{k})^\dagger a_j(\mathbf{k}) d^3k. \tag{24}
\]

Now we can use the Fermi golden rule to calculate the decay rate due to spontaneous emission from an arbitrary initial state \( |i\rangle \) of the atom dressed by the microwave field to any final state \( |f\rangle \), taking into account the density of modes for the photons [31]. The result is

\[
\Gamma = \frac{4\alpha^3 \omega^3}{3} |\langle f | \mathbf{r} | i \rangle|^2, \tag{25}
\]

where \( \omega \) is the energy difference between the initial and final dressed states.

Spontaneous emission will take place between the various states belonging to ladders depicted in Fig. 1. In the process, the projection of the full angular momentum on the \( \hat{O} \hat{z} \) axis is conserved. Here this total angular momentum has three contributions: the atomic angular momentum \( \mathbf{m} \), the angular momentum \( N \) of the driving microwave photons (all in the same mode), and the angular momentum \( M \) in the other modes of the electromagnetic field. Hence we have \( M = N + M = N + m + M = \text{const} \). As spontaneous emission here is considered as a weak perturbation, it will simply induce transitions between eigenstates of the dressed atom [see Eq. (6)], for which \( N \) is a good quantum number (see Sec. II). In the dipole approximation, for a one-photon process, the atomic angular momentum \( m \) can change by at most 1. When the spontaneous photon is emitted in an initially empty mode (which is the process we are interested in), the number of photons in the driving microwave mode \( N \) does not change. Hence, \( N \) changes by at most 1, and the same is true (with opposite sign) for \( M \).

Thus, in the dipole approximation, there are only three possible processes. If the spontaneous photon is emitted with \( \sigma^+ \) polarization, i.e., angular momentum \( M = 1 \), it will induce a transition from a state in a given \( N \) ladder to a state in the \( N = N - 1 \) ladder. Similarly, for \( \sigma^- \) polarization, i.e., angular momentum \( M = -1 \), it will induce a transition from \( N \) to \( N = N + 1 \) and for \( \pi \) polarization, \( M = 0 \) and \( N = N \). As all but one photon modes are initially empty, by spontaneous emission, the energy \( \hbar \omega \) is gained in one mode and thus has to be lost by the dressed atom. Note that here and in the rest of this paper, we discuss the polarization of the various transitions using a quantization axis along the \( \hat{z} \) axis, the direction of the propagation of the driving field. As explained below, the spontaneous photons are mainly \( \sigma^+ \) polarized with respect to the \( \hat{z} \) axis. Hence the helicity of the photons (with respect to their direction of emission, i.e., the quantity measured in an experiment) is in general elliptical, depending on the direction of emission.

In summary, in the dressed-atom picture, as shown in Fig. 1, the only possibilities for emitting a spontaneous photon is to decay to a state with lower energy in the same ladder or in one of the two neighboring ladders. All other processes are forbidden. Note, however that — in the absence of a further approximation — plenty of processes with all 3 possible polarizations should exist. However, as can be seen from Fig. 1, the \( \sigma^+ \) emission is clearly favored, since the \( N - 1 \) ladder is shifted downwards with respect to the initial state by \( \Delta \). \( \pi \) is less favored, and \( \sigma^- \) even less because of the upward energy shift.

### B. Spontaneous emission from a nondispersive wave packet in the harmonic approximation

The general properties discussed in Sec. II are valid whatever initial state is chosen. When the initial state is a nondispersive wave packet, and when the scaled microwave field \( F_0 \) is sufficiently large, we can use the harmonic approximation described above. We will make this approximation throughout the rest of this section. The atomic dipole components \( x = iy, z \) corresponding, respectively, to the photon polarizations \( \sigma_\pm, \pi \), can be expressed as functions of creation and annihilation operators in the normal modes [Eqs. (17) and (18)]: this is straightforward, as all these operators are linear combinations of positions and momenta. The result is

\[
x \pm iy = \Omega^{-2q} \left[ \begin{array}{c}
-3q + Q \mp (q - Q) \frac{\omega_+}{\Omega(1-q)} b_+^\dagger \\
+ \frac{1}{4 \sqrt{\Omega Q}} \left( -3q + Q \pm (q - Q) \frac{\omega_+}{\Omega(1-q)} \right) b_+ \\
-3q + Q \mp (q - Q) \frac{\omega_-}{\Omega(1-q)} b_-^\dagger \\
+ \frac{1}{4 \sqrt{\Omega Q}} \left( -3q + Q \pm (q - Q) \frac{\omega_-}{\Omega(1-q)} \right) b_- \\
\end{array} \right], \tag{26}
\]
Hence, in the harmonic approximation, there are only very few transitions allowed. The selection rules are

\[ \Delta n_x = 0, \quad \Delta n_y = 0, \pm 1, \quad \Delta n_z = 0 \]

or

\[ \Delta n_x = 0, \pm 1, \quad \Delta n_y = 0, \quad \Delta n_z = 0 \]  \hspace{1cm} (28)

in \( \sigma^+ \) and \( \sigma^- \) polarizations, and

\[ \Delta n_x = 0, \quad \Delta n_y = 0, \quad \Delta n_z = \pm 1 \]  \hspace{1cm} (29)

in \( \pi \) polarization.

This means that a given \( |n_x, n_y, n_z; N\rangle \) dressed eigenstate is connected to at most two states in the same \( N \) ladder, and five states in each of the \( N \pm 1 \) neighboring ladders. Among these 12 possible transitions, only those ending in a final state with lower energy are allowed in a spontaneous emission process. Using Eq. (21), we see that the energy loss is \( \omega_+ \) for the two \( \pi \)-polarized transitions, \( \Omega_1 \), \( \Omega_2 \pm \omega_+ \), \( \Omega_3 \pm \omega_- \), for the five \( \sigma^+ \) transitions, and \( -\Omega_1, -\Omega_2 \pm \omega_+ \), and \( -\Omega_3 \pm \omega_- \) for the five \( \sigma^- \) transitions. From Eq. (15), we know that \( \omega_+ \) and \( \omega_- \) are both smaller than \( \Omega \). Hence all five transitions in \( \sigma^- \) polarization are forbidden (by energy conservation) for spontaneous emission. On the other hand, all five transitions in \( \sigma^+ \) polarization are allowed. In \( \pi \) polarization, only the transition \( \Delta n_z = -1 \) is allowed by energy conservation. To summarize, in the harmonic approximation, a given \( |n_x, n_y, n_z\rangle \) atomic eigenstate can only emit spontaneous photons at the six frequencies \( \omega_+, \Omega_1, \Omega_2 \pm \omega_+, \Omega_3 \pm \omega_- \), the first transition being \( \pi \) polarized, the other five having \( \sigma^+ \) polarized.

In the specific case where the initial state is the ground-state wave packet \( |0,0,0\rangle \) with minimum values of the \( n_x, n_y, n_z \) quantum numbers and maximum localization around the equilibrium point, all transitions decreasing any of \( n_x, n_y, \) or \( n_z \) are obviously nonexistent. Hence we are left with only three possible transitions, all of them being \( \pi \) polarized.

(i) The transition from the ground-state wave packet \( |0,0,0; N\rangle \) to its image \( |0,0,0; N-1\rangle \) at microwave frequency \( \Omega \). While the atomic state is not changed, a photon of the driving microwave is elastically scattered into another mode at the same frequency, but with a different direction. In the language of quantum optics, this process can be viewed as the elastic component in the resonance fluorescence of the driven atom.

(ii) The transition from \( |0,0,0; N\rangle \) to a \( n_x = 1 \) dressed state \( |1,0,0; N-1\rangle \) occurring at frequency \( \Omega - \omega_+ \). This is an inelastic component.

(iii) The transition from \( |0,0,0; N\rangle \) to a \( n_z = 1 \) dressed state \( |1,0,0; N-1\rangle \) occurring at frequency \( \Omega + \omega_+ \). This is also an inelastic component. Note that the difference in signs in the frequencies of the two inelastic components, \( \Omega - \omega_+ \) and \( \Omega + \omega_+ \), is directly related to the unusual minus sign in the Hamiltonian equation (20).

In the harmonic approximation, we can go beyond the above qualitative predictions and calculate the partial decay rates to the different final states analytically. To this end, we need the matrix elements of the dipole operators between the initial and final states. This is easily obtained using the expansion of the dipole operator on the creation and annihilation operators in the normal modes [Eq. (27)], as the matrix elements of these operators between eigenstates are well known. We obtain

\[ (0,0,0| -iy|0,0,0) = \Omega^{-2/3} q^{-1/3}, \]

\[ (1,0,0| -iy|0,0,0) = \frac{-3q + Q - (q - Q) \omega_+}{4\sqrt{\Omega Q} N_+}, \]

\[ (0,1,0| -iy|0,0,0) = \frac{3q + Q - (q + Q) \omega_-}{4\sqrt{\Omega Q} N_-}. \]

This finally makes it possible to calculate the decay rates of the three spontaneous emission processes, using Eq. (25):

\[ \Gamma(\Omega) = \frac{2\alpha^2 \Omega^{2/3} q^{-2/3}}{3} \]

for the elastic component, and

\[ \Gamma(\Omega - \omega_+) = \frac{2\alpha^2 (\Omega - \omega_+)^3}{3}|(1,0,0| -iy|0,0,0)|^2 \]

and

\[ \Gamma(\Omega + \omega_+) = \frac{2\alpha^2 (\Omega + \omega_+)^3}{3}|(0,1,0| -iy|0,0,0)|^2 \]

for the two inelastic components. The various quantities in these formula are related to the microwave frequency \( \Omega \) and to the scaled parameter \( q = \frac{\text{it}}{\text{energy}} \) related to the microwave amplitude by Eqs. (10) and (11) by Eqs. (15) and (32). For a fixed classical dynamics (fixed \( q \) i.e., fixed scaled microwave amplitude \( F_0 \)), the elastic decay rate scales as \( \Omega^{5/3} = n_0^{-5} \), as does the decay rate of a circular state while the inelastic decay rates scale differently as \( \Omega^2 = n_0^{-6} \). The results can be converted to usual units by multiplying them by \( me^4/\hbar^3 = 4.13 \times 10^{16} \text{s}^{-1} \).

In Fig. 2, we show the frequencies, squared dipole matrix elements and decay rates (in linear and logarithmic scales) of the three components as a function of the scaled microwave field for principal quantum number \( n_0 = 60 \), the corresponding microwave frequency being \( \Omega/2\pi = 30.46 \text{ GHz} \). The first thing to note is obviously the dominance of the elastic component at the microwave frequency. About 80% of the spontaneous photons emitted by the system are at the microwave frequency. Elastic scattering of the microwave photons to other modes (different direction) is by far the dominant process. The radiation diagram of these photons has the standard
dipolar shape (this is identical for the three components as they are all due to the $x-iy$ dipole operator). This dominance is easily understood from the different scalings of the three decay rates, Eqs. (33)–(35), discussed above. The dipole matrix element $(0,0,0|x-iy|0,0,0)$ measures the static dipole, i.e., the position of the center of the nondispersive wave packet. It is thus — in the harmonic approximation — equal to $x$, the coordinate of the classical fixed point, of the order of $n_0^2$. In contrast, the nondiagonal matrix elements $(0,1,0|x-iy|0,0,0)$ or $(1,0,0|x-iy|0,0,0)$ measure how the excited states explore the vicinity of the fixed point. They are of the order of $n_0^{3/2}$, i.e., about $\sqrt{n_0}$ times smaller. This is why the corresponding decay rates are much smaller. Because of the different scaling properties of the decay rates, it is easy to check that a change in the value of $n_0$ gives similar plots for the decay rates of the inelastic components as a function of the scaled microwave field (they are just multiplied by a uniform numerical factor) while the elastic component is relatively increased in proportion with $n_0$. In other words, in the semiclassical limit $n_0 \to \infty$, the inelastic components asymptotically vanish.

Another important observation is that, although the two inelastic components have dipole matrix elements of comparable magnitude [see Fig. 2(b)], the decay rate of the $\Omega-\omega_+$ component is several orders of magnitude smaller than the decay rate of the $\Omega+\omega_-$ component; see Fig. 2(c). This is entirely due to the $\omega^2$ factor representing the density of electromagnetic modes. In practice, the $\Omega-\omega_+$ component will be very difficult to detect.

Finally, when the microwave field is sufficiently small, the various quantities can be expanded in powers of the scaled field. In this limit [see also Fig. 2(d)], the $\Omega-\omega_+$ decay rate goes to zero as $F_0^3$, whereas the $\Omega+\omega_-$ decay rate increases like $1/\sqrt{F_0}$ and the elastic component $\Omega$ keeps a constant decay rate. Of course, such a divergence is clearly unphysical, and simply indicates the breakdown of the harmonic approximation. At lower field, the pendulum approximation of the Hamiltonian [21] has to be used, restoring finite decay rates, roughly equally shared by the $\Omega+\omega_-$ and $\Omega$ components.

IV. COMPARISON WITH EARLIER RESULTS

A former approach to spontaneous emission of nondispersive wave packets [7] was carried out in the rotating frame using the Hamiltonian equation (3), or rather its generalization, which also includes other nonoccupied electromagnetic modes. Using the notion of rotational frequency shift found previously by the same authors [13], the frequency of the emission in the laboratory frame has been found. This is a single frequency, since the authors considered the emission at $\Omega+\omega_-$, assuming only, incorrectly, that spontaneous emission of a photon in the rotating frame must lead to a decrease of the energy (as it happens in the laboratory frame).

To see the fallacy of this argument, consider the standard spontaneous decay of the circular $|n+1,n,n\rangle$ hydrogenic state in the absence of any driving electromagnetic field. In the dipole approximation, it can decay spontaneously only to the $|n,n-1,n-1\rangle$ circular state. Consider the same situation in the frame rotating at frequency $\Omega$. The Hamiltonian in the rotating frame is obtained from Eq. (3) with $\Omega=0$. The energy of the initial state in the rotating frame is then $E_f=-1/2(n+1)^2-\Omega n$, while the energy of the final state $E_f=-1/2n^2-\Omega(n-1)$ is greater than $E_f$, provided $\Omega>(n+1)/2(n+1)$, provided $\Omega>(n+1)/2(n+1)$. Hence emission of the spontaneous $\sigma^+$-polarized photon increases the energy in the rotating frame.

The same situation occurs for the standard toy model of quantum optics: the two-level atom interacting with a qua-
sireonat monochromatic field, treated in the rotating-wave approximation. In the “rotating frame” [23], spontaneous emission does not occur only from the upper to the lower state of the (Autler-Townes) doublet. This particular line (upper to lower state in the doublet) occurs at the frequency of the driving field plus the Rabi frequency. But the emission may also take place from the lower to the upper state of the doublet (at driving frequency minus the Rabi frequency), as well as from the lower (upper) state to “itself.” In effect, the resonance fluorescence spectrum has the famous Mollow triplet structure [32,33], with the dominant elastic component in the middle.

This discussion shows that the rotating frame approach leads to quite a counterintuitive picture. By comparison, the dressed-state picture presented in Sec. III seems to be quite intuitive. One may easily verify that extending the calculation presented in Ref. [7] to include the lines emitted at the microwave frequency \( \Omega \) (which corresponds in the rotating frame to the emission from the wave packet state to itself), as well as at \( \Omega - \omega_+ \), leads to the same results as those obtained in the dressed-atom picture above [33]. This shows that both approaches lead, as they should, to the same spontaneous emission rates.

From a practical point of view, as the decay at \( \Omega - \omega_+ \) is much slower than the decay at \( \Omega + \omega_+ \) (see the discussion above), the numbers given in Ref. [7] can be considered correct [34] if elastic scattering does not have to be taken into account. A nontrivial question is whether elastic scattering of photons has to considered as a decay process or not. In our opinion, this is simply a question not properly formulated. Thinking of resonance fluorescence in terms of decay processes will come into the game as it destroys the phase coherence of the initial wave packet and consequently broadens the line. A general discussion of such effects, and the general theory of resonance fluorescence in the framework of quantum optics, can be found in Ref. [32].

Let us also note that the “classical” emission rate of an electron moving on a circular orbit presented in Ref. [7] is incorrect. A classical charge moving on a circular orbit with frequency \( \Omega \) must emit photons at the same frequency. In our opinion, the only meaningful quantity is the decay rate of the energy in the atomic system. Classically, it is proportional to the square of the acceleration and, for the classical periodic orbit at the microwave frequency, it is in atomic units [35]:

\[
\frac{dE}{dt} = 2\alpha^3 \Omega^{8/3} q^{-2/3}.
\] (36)

Considering that, in the semiclassical limit \( n_0 \rightarrow \infty \), the system radiates essentially photons as frequency \( \Omega \), we immediately see that this classical rate exactly (i.e., without an additional numerical factor of the order of one) coincides with the quantum decay rate of energy deduced from Eq. (33). It also exactly corresponds to the decay rate of atomic circular states, as given in, e.g., Ref. [36], in the limit of vanishing microwave field \( q \rightarrow 1 \). In Ref. [7], the decay rate in the \( \Omega + \omega_+ \) component was incorrectly compared to the relative decrease of the classical energy in time. The first quantity — as discussed above — is roughly \( n_0 \) times smaller than the elastic decay rate, while the relative decrease of classical energy in time is also about \( n_0 \) times smaller than the rate of emission of photons because the energy of a radiated photon is about \( n_0 \) times smaller than the total energy. The two \( n_0 \) dependences cancel out, but not the numerical factors in front. This explains why a false classical-quantum correspondence (with constant numerical factors) was observed in Ref. [7]. Both classical and quantum decay rates are — in our opinion — underestimated by a factor \( \approx n_0 \).

V. HOW GOOD IS THE HARMONIC APPROXIMATION?

From the beginning of investigations on nondispersive wave packets in the hydrogen atom driven by a circularly polarized microwave, there has been controversy regarding the accuracy of the harmonic approximation used when introducing the wave packets as Gaussians centered around the equilibrium point [1]; see Sec. II. First, in Ref. [15], and later in a number of papers [16–20], Farrelly and co-workers claimed that “highly anharmonic nature of the effective potential in the vicinity of the equilibrium point will cause any such putative coherent state to disperse” unless the wave packet is built from states of unrealistically high quantum number (\( n_0 \gtrsim 200 \)). They showed that an additional static magnetic field applied in the direction of microwave propagation (\( O_x \) axis) allows one to relatively decrease the importance of the anharmonic terms in the expansion around the equilibrium point. Further, they claimed that “because these extrema are locally harmonic, their vacuum states are truly coherent states in the original sense of Schrödinger” [20] (by extrema one should understand global equilibrium points).

Our understanding of the phenomenon is slightly different. While Farrelly and co-workers stressed that the Gaussian shape of the wave packet is essential for nonspreading time evolution in the locally harmonic potential, we adopt a bit less narrow-minded point of view. We have shown [4] that, even if the harmonic approximation is only partially valid, there will be a one-to-one correspondence between the quasienery states of the approximate harmonic Hamiltonian [Eq. (20)], and the full Hamiltonian in the rotating frame [Eq. (3)]. In particular, the ground state of the local harmonic approximation (referred to above as the ground-state wave packet) is a good approximation of an exact eigenstate of the full problem (or via the unitary transformation to the corresponding Floquet or dressed state). We refer to this exact eigenstate of the full atom plus microwave field problem as the nondispersive wave packet since it is a linear combination of atomic states, and it is centered at the stable equilibrium point in the rotating frame, and thus follows the classical circular periodic orbit of the problem in the laboratory frame; thus the eigenstate fulfills standard textbook criteria for “a wave packet.” Being an eigenstate of \( H_n \) [Eq. (3)], the state is rigorously nonspreading, provided we neglect the
coupling to the continuum — i.e., the possibility of ionization. We have shown [4–6] that typical lifetimes for the ionization exceed many thousands of Kepler periods (reaching up to $10^6$ Kepler periods for $n_0 = 60$ or more).

In our understanding of the phenomenon, the validity of the harmonic approximation is not crucial for the very existence of nondispersive wave packets in a hydrogen atom driven by a circularly polarized microwave. Moreover, the phenomenon is more general and, as mentioned in Sec. I, may be found in other systems such as atoms driven by a linearly polarized microwave [2] or one-dimensional models [25] supporting stable classical 1:1 resonance islands. In the general case, the wave packet preserves its shape at intervals equal to the period of the external driving: only in the circular polarization problem (due to the additional symmetry which allows for the complete removal of the time-dependence) is the wave-packet shape the same at all times. There is no reason to restrict the notion of a nondispersive (or nonspreading) wave packet to a single particular situation which can be made almost fully harmonic. One should also remember that even in the presence of a magnetic field, a case favored by Farrelly and co-workers, the potential is also only approximately harmonic, and Gaussian wave packets will eventually disperse. Wave packets in our sense, being single dressed states, will not. In any case, while the addition of a magnetic field does not change the nature of the problem, it may just keep some anharmonic terms smaller.

While not necessary for the very existence of nondispersive wave packets, the harmonic approximation is useful for an analytic evaluation of different properties, such as the spontaneous emission discussed above. In view of the remarks of Farrelly and co-workers, it would seem that the results obtained in the Sec. IV are incorrect. The simplest way to check this statement is to evaluate the “exact” spontaneous emission spectrum from the ground quasienergy eigenstate of the full Hamiltonian.

The presence of anharmonic terms in the full Hamiltonian has to manifest itself in the breakup of the selection rules discussed in Sec. II. Instead of three possible lines in the $\sigma^+$ emission, one should expect a multitude of lines. Using the harmonic level assignment, the $\{0,0,0; N\}$ wave packet may then decay to, e.g., states approximated by $|i,j,k; N-1\rangle$, with $i,j,k$ being larger than unity (with $k$ even, since $\sigma^+$ dipole element may link states of same $z$ parity only). Also, due to couplings introduced by nonlinear terms, spontaneous emission with other (i.e., $\sigma_-$ or $\pi$) polarizations becomes possible.

Thus the spontaneous emission spectrum provides a useful test of the validity of the harmonic approximation. Having at our disposal a computer code which reliably allows us to find the exact quasienergy states of the problem [4–6], it has been sufficient to evaluate dipole matrix elements between the exact quasienergy states. The resulting squared dipole moments are shown in Fig. 3 together with the corresponding squared dipoles as given by the harmonic approximation [Eq. (32)], presented as three crosses. Observe that the harmonic approximation works quite well. The selection rules are well preserved, the frequencies of the emission (line positions) agree very well, and the dipole values (heights of the lines) agree within 10% for all three lines. This shows that the harmonic approximation is quite good, in the absence of any magnetic field, not only for the ground-state wave packet $|0,0,0\rangle$, but also for primary $\omega_+$ [state $|1,0,0\rangle$] and $\omega_- [state |0,1,0\rangle]$ excitations. Observe that the height of the peak at $\Omega$ is proportional to $|\langle 0,0,0| x-iy | 0,0,0 \rangle|^2$ involving the wave function of the ground-state wave packet only. The sidebands, corresponding to transitions to excited wave packets, provide independent tests of the corresponding wave functions.

Figure 4 shows the corresponding spontaneous emission rates. The component at $\Omega - \omega_+$ is not visible due to the low frequency of the emission. On the other hand, the spectrum at high frequencies is enhanced by cubic power of the frequency factor coming from the photonic density of states. Here careful inspection of the spectrum corresponding to exact eigenstates reveals a tiny peak centered at $\Omega + 2\omega_-$ (indicated by an arrow in the figure). Such a transition is forbidden in the harmonic approximation; its presence indicates that the harmonic approximation is not exact. Still, the smallness of the anharmonic terms strongly shows that the harmonic approximation works well, at least for the typical value $n_0 = 60$.

If one considers a lower value of $n_0$ (say around $n_0 = 30$) when the effective $\hbar$ is larger, the anharmonic terms in
Similarly to the case shown in Figs. 3 and 4, provided no done using, for example, spherical coordinates, similar calculations can be consistently done at the lowest order of the harmonic approximation — the wave function in spherical coordinates is a better approximation of the “exact” wave function than the corresponding one in Cartesian coordinates, as it incorporates some of the anharmonic terms; the price to pay is the breaking of the selection rules [Eqs. (28) and (29)]. Hence the transition rates calculated using harmonic wave functions in spherical coordinates already incorporate some of the higher-order terms, and might be closer to the “exact” results. Note, however that if the calculation is performed consistently at any order in perturbation theory (the anharmonic terms being the perturbation), the results should of course be independent of the choice of coordinates.

VI. CONCLUSIONS

It is worth stressing again that, in the semiclassical limit (small $\Omega$ or large $n_0$), the dominant contribution to the spontaneous emission comes from the elastic component at frequency $\Omega$, which does not destroy the wave packet but rather leads to the decoherence in its time evolution. The wave packet converts the energy of the driving photon into a spontaneously emitted photon, moving at the same time on the circular orbit.

From a practical point of view, the spontaneous emission rates are several orders of magnitude smaller than the ionization rate in a typical situation $n_0=60$ and $F_0=0.05$ [6], and thus unlikely to be observable. However, this ratio can be made smaller by increasing $n_0$. Indeed, as shown in Ref. [6], the typical ionization rate decreases exponentially with $n_0$ (because it is due a chaos-assisted tunneling process) while the spontaneous decay rates decrease algebraically with $n_0$. For very large $n_0$, spontaneous emission will eventually become the dominant process. For $F_0=0.05$, the crossover should take place around $n_0=200$, which makes it very difficult to observe experimentally. Another interesting possibility is to decrease the scaled field $F_0$ so that the ionization rate drops considerably. According to numbers shown in Ref. [6], the crossover for $n_0=60$ should take place around $F_0=0.03$ in a regime where the harmonic approximation is still valid (the results are not shown, since they are very similar to those depicted in Fig. 4). In any case, the total decay rate is of the order of a few tens of photons per second, making the experimental observation rather difficult.

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We shall not consider typical quantum-mechanical effects important when the number of photons in the occupied mode is small, such as quantum revivals. The equivalence between dressed-state and Floquet pictures is obtained, let us recall, by introducing a different static Hamiltonian in the rotating frame, and hence different creation and annihilation operators for the normal modes. However, the energy levels of the dressed atom — and consequently the frequencies of the normal modes — do not depend on the choice of gauge. The same is true for the decay rates. The “exact” computations shown in this paper have been performed in the velocity gauge.

The calculation is here done for the density of electromagnetic modes in free space. The decay rates can be substantially modified in the presence of a cavity, as could be used in a real experiment [8].

In Ref. [7], the matrix elements of the dipole operator are calculated using explicit forms of the wave functions in configuration space. We prefer to use no explicit representation of the eigenstates and manipulate creation and annihilation operators in the normal modes: extension of the present results to the decay of “excited” nondispersive wave packets is much easier this way.

Note, however, that the spontaneous decay rate per Kepler period as given in Ref. [7] at the end of p. 3630 has a $2\pi$ factor wrongly placed. It is thus “only” four orders of magnitude smaller than the ionization rate.

In the context of quantum optics of a two-level system, the rotating wave approximation makes it possible to define a time-dependent unitary transformation producing a static Hamiltonian. As there is a formal analogy, this is known as passing to the “rotating frame,” although this is in general not a purely geometrical transformation. Only in the case of a circularly polarized field do the two acceptions of “rotating frame” coincide.

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