Breakdown of correspondence in chaotic systems: Ehrenfest versus localization times

Zbyszek P. Karkuszewski,1,2 Jakub Zakrzewski,2 and Wojciech H. Zurek1
1Theoretical Division, T6, MS B288, LANL, Los Alamos, New Mexico 87545
2Instytut Fizyki, Universytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland
(Received 27 March 2001; published 10 April 2002)

Breakdown of quantum-classical correspondence is studied on an experimentally realizable example of a one-dimensional periodically driven system. Two relevant time scales are identified in this system: the short Ehrenfest time $t_h \approx -\ln(\hbar^{-1})$ and the typically much longer localization time scale $T_L \approx \hbar^{-2}$. It is shown that surprisingly weak modification of the Hamiltonian may eliminate the more dramatic symptoms of localization without effecting the more subtle but ubiquitous and rapid loss of correspondence at $t_h$.

DOI: 10.1103/PhysRevA.65.042113
PACS number(s): 03.65.Sq, 05.45.Mt, 03.65.Ta

Quantum realizations of classically chaotic systems behave differently than their fully classical counterparts. The breakdown of quantum-classical correspondence occurs on two distinct time scales: the short Ehrenfest time

$$t_h \approx \frac{1}{\lambda} \ln \left( \frac{\Delta p \chi}{\hbar} \right) \approx \frac{1}{\lambda} \ln \left( \frac{\Delta A}{\hbar} \right)$$

(1)

gives the time scale at which the quantum minimal wave packet spreads sufficiently over a macroscopic part of the phase space to feel nonlinearities in the potential [1–3]. In Eq. (1) $\Delta p$ denotes the initial uncertainty in momentum, $\lambda$ is the Lyapunov exponent, $\chi \equiv \sqrt{\partial_x V \partial_p^2 V}$ is a typical scale on which nonlinearities in the potential $V$ are significant and $A$ is the classical action. Ehrenfest time $t_h$ marks the moment when quantum corrections become important in the time evolution of the Wigner function $W(x,p)$, governed by the Wigner transform of the von Neumann bracket, known as the Moyal bracket [4]

$$\frac{d}{dt} W = \{H,W\}_{MB}$$

$$= \{H,W\} + \sum_{n=1}^{\infty} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} \frac{\partial^{2n+1} V}{\partial x^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}.$$  

(2)

The Moyal bracket appearing above is defined as

$$\{H,W\}_{MB} = -\frac{2}{\hbar} H \sin \left( \frac{\hbar}{2} \left( \frac{\partial}{\partial p} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial p} \right) \right) W.$$  

The arrows indicate the direction of differentiation.

In a chaotic system initially localized distribution will be stretched in the unstable directions. In a Hamiltonian system, this stretching must be matched by shrinking in the stable directions as mandated by the Liouville theorem. Consequently, the typical width of the effective support of classical probability distribution shrinks as $\exp(-\lambda t)$. That leads to an exponential increase of higher-order derivatives of $W$ in Eq. (2), so after $t_h$ quantum corrections become comparable to terms given by the Poisson bracket [3].

There is evidence that at $t_h$ also the averages over the phase space show symptoms of correspondence loss [5,6] although not everyone agrees [7,8]. The first goal of our paper is then to investigate the dependence of the onset of the loss of correspondence in the averages on $\hbar$, and to show that it is consistent with Eq. (1).

In addition to the rapid correspondence loss on the logarithmic time scale $t_h$ there is plentiful evidence for the dynamical localization in a class of quantum chaotic systems [9–11]. It sets in on a much longer localization time scale

$$T_L \approx \frac{d}{\hbar^2},$$

(3)

where $d$ is a diffusion constant characterizing initial growth of variance in momentum of the quantum system. Tutorial example of localization is described in [12]. It is of obvious interest to consider systems in which both mechanisms of correspondence loss are present, and to investigate their behavior with varying parameters. This is the second goal of our paper. The system we investigate is a straightforward modification of an experimentally studied system based on an optical lattice [11]. Therefore, it should be simple to verify our predictions in the laboratory.

We study quantum-classical correspondence in a system described by the Hamiltonian written in dimensionless variables

$$H(x,p) = \frac{p^2}{2m} - \kappa \cos(x - \ell \sin t) + a \frac{x^2}{2}.$$  

(4)

For $a=0$, this Hamiltonian reduces to a model recently investigated both theoretically [13–15] and experimentally [11] corresponding to a motion of cold atoms in a quasiresonant standing wave with periodically moving nodal pattern. Classical behavior of the system is of the mixed type with large regions of phase space dominated by a chaotic motion (e.g., Lyapunov exponent of the chaotic component is $\lambda = 0.2$ for chosen parameter values $m = 1$, $\kappa = 0.36$, $l = 3$, $a = 0$ [14]).

To investigate quantum-classical correspondence one can start by looking at the Wigner function. Structures which appear in $W$ are known to be different from (but not unrelated to) the classical probability distribution in the phase...
FIG. 1. (Color) Difference between classical, $\Delta_2^{\text{cl}}$, and quantum, $\Delta_2^{\text{qm}}$, variances for three values of $\hbar$ and for the same initial condition, i.e., coherent state centered at $(0,0)$ in the phase space. In the inset, $\tau_{\text{EH}}$ dependence on $\ln(1/\hbar)$ averaged over few initial conditions is shown. Error bars indicate standard deviation of the results for a given value of $\hbar$. Criterion used to estimate Ehrenfest time scale compares magnitude of Poisson bracket and a quantum correction (the rest of the Moyal bracket in (2)).

FIG. 2. (Color) Difference between classical and quantum variances for $\hbar = 0.0256$ and four initial conditions [coherent states centered at points $(0,0)$, $(\pi/2,0)$, $(-\pi/2,0)$, and $(\pi,0)$ in the phase space] is plotted using black, red, green, and blue lines, respectively. The inset shows the discrepancy between classical and quantum averages as defined by Eq. (5) as a function of $\hbar$. The dependence seems to be logarithmic.
space. One can especially note the existence of the interference fringes that saturate on small scales associated with action \( s \sim (2\pi\hbar)^2/S \), where \( S \) is the action of the system — e.g., the phase-space volume over which \( W \) can spread [16].

One might nevertheless suppose that the quantum structures appearing on such small scales in the Wigner function will have little effect on the averages. In particular, averages typically concern large scales, and, therefore, should not be directly affected by small-scale interference. This is, however, only partially true.

Figure 1 shows the difference between classical \( \Delta_q^2 \) and quantum \( \Delta_{qm}^2 \) averages of a typical quantity \( \langle \Delta^2 = (p^2) - \langle p \rangle^2 \rangle \) for \( \hbar = 0.16, 0.064, \) and 0.0256 obtained in the time evolution resulting from Hamiltonian (4) with \( \alpha = 0 \). Initial classical distribution is a two-dimensional Gaussian in phase space and corresponds to a Gaussian wave packet used for quantum evolution. The loss of correspondence is apparent: early on, a noticeable difference between the two averages develops. The smaller the value of \( \hbar \), the later it appears. The inset shows that the discrepancies occur after a time which seems indeed logarithmic in \( \hbar \). They persist without a significant change in their magnitude until logarithmic localization begins to dominate. Vertical error bars in the insert arise from the quantum and classical averages of a typical quantity \( \langle \Delta^2 = (p^2) - \langle p \rangle^2 \rangle \) for various initial conditions at given \( \hbar \). This in turn affects the value of \( t_h \) (see Fig. 2).

It is intriguing to enquire about the magnitude of the typical discrepancy between the quantum and classical averages in the time interval between \( t_h \) and \( T_L \). To address this question we have evaluated

\[
\delta(h) = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} |\Delta^2_q - \Delta^2_{qm}| dt,
\]

for \( t_h < t_1 < t_2 < T_L \) as a function of \( \hbar \). The results (averaged over several initial conditions) show that \( \delta(h) \) decreases with decreasing \( \hbar \) significantly slower than linearly (compare the inset in Fig. 2). While our data do not allow for a definite conclusion, they point out towards a logarithmic \( \hbar \) dependence of \( \delta(h) \). The results obtained for \( \delta(h) \) depend slightly on the choice of \( t_1 \) and \( t_2 \). That, together with the size of the error bars, does not allow us to be more definitive. The logarithmic dependence is not in disagreement with results of [17] [who claim that largest deviations are \( O(\hbar) \)] obtained for a model of two interacting spins. Our data indicate a similar behavior for a realistic, experimentally accessible system. Still, the latest result for the interacting spin system [18] gives a square root rather than the logarithmic behavior.

To the best of our knowledge, \( t_h \) has not yet been detected experimentally. On the other hand, it is known that random external noise [19,20] as well as decoherence [3,6,21,22] suppress evidence of the breakdown at \( t_h \) or at \( T_L \), restoring quantum-classical correspondence. This happens when the coherence length [22,23]

\[
l_c = \hbar \sqrt{\lambda/2D}
\]

is maintained by the chaotic system in the presence of decoherence [which can be parametrized under certain conditions [22,23] by the addition of the diffusive term \( -D \beta^2 \dot{W} \) to the evolution Eq. (2)] becomes small compared to the scale of nonlinearities \( \chi \),

\[
l_c < \chi.
\]

The price for the restoration of quantum-classical correspondence by decoherence is irreversibility [3,22].

Let us now consider the localization. At the onset of localization (clearly visible in Fig. 3 for \( a = 0 \)) the difference between the classical and quantum variances begins to rapidly increase, in a manner approximately consistent with \( \Delta^2_q - \Delta^2_{qm} \propto d(t-T_L) \). This is to be expected. In systems affected by localization, wave functions exhibit characteristic exponential form, as exemplified in Fig. 4(a).

Localization can be suppressed by noise and decoherence, as was recently shown by two groups using similar experimental setups [23–25]. Figure 3 shows that a much simpler mechanism for suppression of localization exists: When the driven periodic potential is supplemented by even a weak harmonic potential, [controlled by the parameter \( a \) in Eq. (4)] localization disappears while chaotic character of motion persists, as illustrated by Poincaré surfaces of section in Fig. 5.

In the presence of even a weak harmonic force a prominent growth of the difference between quantum and classical averages nearly disappears. Also wave functions cease to be exponentially localized—compare Fig. 4(b). At first glance, one may find this dramatic response to even a weak harmonic potential quite surprising.

Our explanation of this effect is simple and general. Numerical studies show that the suppression of localization occurs when a quarter of the period of the harmonic oscillator is approximately the localization time scale \( T_L \). Localization becomes maintained only when the “swapping” between \( x \) and \( p \) generated by the harmonic evolution happens on a time scale longer than \( T_L \). (After all, localization in \( p \) is incompatible with localization in \( x \), since the wave function cannot have the exponential form in both complementary observables.) In fact, we have seen disappearance of localization when the period of the harmonic oscillator is about 16\( T_L \).

So, the “swapping” does not need to be very frequent. This simple explanation of suppression of localization is model independent and may be the key result of our paper.

Experimental verification of our two key results concerning (i) \( t_h \) and (ii) \( T_L \) should be possible. It will be more difficult to check for the intriguing prediction of the correspondence breakdown on short time scale \( t_c \). One would ideally want to have two otherwise identical systems, one of them quantum, the other classical. In our quantum universe this is unfortunately impossible. One remedy would be then to compare real experimental results with the classical computer experiment for the same values of all relevant parameters.

There is, however, something disingenuous about comparing a classical computer simulation with a quantum experiment in an attempt to find a breakdown of quantum-classical...
correspondence. We shall therefore suggest an alternative strategy: Effective value of $\hbar$ can be adjusted in the experiments [11,23]. Therefore, it may be possible to look for differences between some average quantities, say $D_2$, for two otherwise identical systems which are quantum to a different extent that is adjusted by selecting different values of $\hbar$. It is obvious from Fig. 1 that significant differences between two systems endowed with different effective value $\hbar$ would appear, and it should be possible to check for the validity of Eq. (1).

Destruction of localization by introduction of the harmonic trap into experiments which test quantum chaos in optical lattices should be much easier to verify. It may be desirable to superpose optical lattice of the sort used to investigate quantum chaos [11,23] in a magnetooptical trap anyway (e.g., to confine the system such as Bose-Einstein condensate, where investigation of chaos may be of interest in its own right [26]). It should be then relatively easy to see if our prediction of the loss of localization is indeed borne out by the experiments.

Both of our key conclusions, while investigated numerically in a specific system selected for its relevance to experiment should have a broad validity. In particular, destruction of localization due to rotation of phase space as a whole by the background (e.g., harmonic) potential on a time scale comparable to the localization time $T_L$ is justified by a simple physical argument. This effect is dramatic and should

FIG. 3. (Color) Difference between classical and quantum variance for four values of parameter $a$, as indicated in the figure, and for $\hbar = 0.16$. Addition of harmonic potential destroys dynamical localization present for $a = 0$. All initial conditions are the same.

FIG. 4. Time-average probability density in momentum representation: (a) exponentially localized for $a = 0$, (b) Gaussian for $a = 0.01$.

FIG. 5. Stroboscopic Poincaré surfaces of section for the system with parameter $a = 0$ (left), and for $a = 0.01$ (right).
be easily accessible to experiments. The earlier and more subtle phenomena that occur at $t_0$ are robust to the addition of an overall potential, providing that the chaotic character of the system is preserved. Our suggestion for investigation of the phenomena relies on the ability to adjust the relevant effective $\hbar$ in experiments. Moreover, we have found indications that such quantum-classical discrepancies in the averages diminish more slowly than $O(\hbar)$, which — if confirmed in experiments we suggest — would be of major importance to our understanding of the classical limit of quantum theory.

J.Z. and Z.P.K. acknowledge support by KBN under Grant No. 2P03B00915, and W.H.Z. was supported in part by NSA.