GEOMETRY OF QUANTUM ENTANGLEMENT

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Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states

Space of pure states for an arbitrary $N$:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N - 2$ real dimensions.
Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}(|\psi\rangle, |\varphi\rangle) := \arccos |\langle \psi | \varphi \rangle|$$

Unitary evolution

Let $U = \exp(iHt)$. Then $|\psi'\rangle = U|\psi\rangle$.

Since $|\langle \psi | \varphi \rangle|^2 = |\langle \psi | U^\dagger U |\varphi\rangle|^2$ any unitary evolution is an isometry (with respect to any standard distance!)

Quantum Chaos: what happens for large $N$?

How an isometry may lead to classically chaotic dynamics?

The limits $t \rightarrow \infty$ and $N \rightarrow \infty$ do not commute.
Mixed quantum states

Set $\mathcal{M}_N$ of all mixed states of size $N$

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \text{Tr}\rho = 1\}$$

example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$$\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$$
where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

How the set of all $N = 3$ mixed states looks like?

An 8 dimensional convex set with only 4 dimensional subset of pure (extremal) states, which belong to its 7 dim boundary.
Euclidean Geometry of the set $\mathcal{M}^{(N)}$ of quantum states with respect to **Hilbert–Schmidt** distance:

$$D_{\text{HS}}(\rho, \sigma) := \sqrt{\text{Tr}((\rho - \sigma)^2)}$$

Set $\mathcal{M}^{(N)}$ can be inscribed into an outsphere centred at the maximally mixed state $\rho_* = 1/N$ of radius $R_N = \sqrt{(N - 1)/N}$. The insphere inscribed inside $\mathcal{M}^{(N)}$ has radius $r_N = 1/\sqrt{(N - 1)N}$, so $R_2 = r_2$.

**Hilbert–Schmidt distance** leads to HS (flat) measure. For $N = 2$ (one–qubit states) we receive:

- the volume of the Bloch ball $V(2) = \pi\sqrt{2}/3$ and
- the area of the Bloch sphere of radius $R_2 = 1/\sqrt{2}$ reads $A(2) = 2\pi$. 
Volume and area of the set $\mathcal{M}^{(N)}$ of mixed states

with respect to the HS measure

**Volume in $N^2 - 1$ dimensions**

$$V(N) = \text{vol}(\mathcal{M}^{(N)}) = \sqrt{N} (2\pi)^{N(N-1)/2} \frac{\Gamma(1) \cdots \Gamma(N)}{\Gamma(N^2)} ,$$  \hspace{0.5cm} (1)

**Hypersurface area of this $N^2 - 2$ dimensional boundary**

$$A(N) = \text{vol}(\partial \mathcal{M}^{(N)}) = \sqrt{N - 1} (2\pi)^{N(N-1)/2} \frac{\Gamma(1) \cdots \Gamma(N + 1)}{\Gamma(N)\Gamma(N^2 - 1)} .$$  \hspace{0.5cm} (2)

**Area/volume ratio $\gamma = Ar/V$**

$$\gamma(\mathcal{M}^{(N)}) = r_N \frac{A}{V} = \frac{1}{\sqrt{(N - 1)N}} \sqrt{N(N - 1)(N^2 - 1)} = N^2 - 1 = d .$$  \hspace{0.5cm} (3)

**Convex Body** = a convex, compact set in $\mathbb{R}^d$

A body $X$ has a constant height if

a) every boundary point of $X$ is contained in a face tangent to the inscribed ball of radius $r$ $\iff$

b) area/volume ratio is fixed, $\gamma(X) := rA/V = d = \dim(X)$.

Archimedean formula for the volume of a $d$-cone:

Volume $V = Ah/d$, where $A$ stands for the area of its base and $h$ for its height $\implies$

A body of a constant height can be decomposed into cones of the same height!

Set of mixed quantum states has a constant height
Composed systems & entangled states

**bi-partite systems:** $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

**Two–qubit system:** $N = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$

**Entanglement measures**

For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \text{Tr}_B |\psi\rangle \langle \psi|$.  

**Definition:** Entanglement entropy of $|\psi\rangle$ is equal to von Neuman entropy of the partial trace

$$E(|\psi\rangle) := -\text{Tr} \sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...
Entanglement of two real qubits

Entanglement entropy at the tetrahedron of $N = 4$ real pure states
Two qubits: foliation of complex projective space \( \mathbb{C}P^3 \) into leaves of equal FS distance to the set of product states \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).

A generic 5–D leaf of an intermediate entanglement \( \omega = C/2 > 0 \) has the local structure of \( U(2)/[U(1)]^2 \times \mathbb{R}P^3 \).
The maximal ball inscribed into $\mathcal{M}^{(4)}$ of radius $r_4 = 1/\sqrt{12}$ centred at $\rho_* = \mathbb{1}/4$ is separable!

K. Ž, P. Horodecki, M. Lewenstein, A. Sanpera, 1998
Two-qubit mixed states

Degree of entanglement: a distance to the closest separable state

(entranglement of formation)

K.Ž, M. Kuš, 2001
The set of separable states of two–qubit system arises as an intersection of $\mathcal{M}^{(4)}$ and its mirror image with respect to partial transposition $T_A(\mathcal{M}^{(4)})$.

The set of two–qubit separable states has a constant height!

S. Szarek, I. Bengtsson, K.Ż, 2006
In 2005 P. Slater studied numerically the ratio $\Omega$ between the probabilities of finding a PPT state in the interior of $\mathcal{M}(N^2)$ and at its boundary,

$$\Omega \equiv \frac{p_V}{p_A} := \frac{V_{\text{PPT}}/V_\text{tot}}{A_{\text{PPT}}/A_\text{tot}} = \frac{V_{\text{PPT}}}{V_\text{tot}} \frac{A_\text{tot}}{A_{\text{PPT}}}.$$  \hspace{1cm} (4)

Area of the PPT part of the boundary of the set of mixed states

$A_P = A_{\text{PPT}}/2$ due to reflection symmetry. Since the set of PPT states has the same constant height $r$ as the set of all states then

$rA_\text{tot}/V_\text{tot} = rA_{\text{PPT}}/V_{\text{PPT}}$. Hence we find

$$\Omega = \frac{V_{\text{PPT}}}{V_\text{tot}} \frac{A_\text{tot}}{A_{\text{PPT}}/2} = 2.$$ \hspace{1cm} (5)

This statement concerns the PPT property for any $N \times N$ system, and for two–qubit system it concerns also the relative probability of finding a separable state.

S. Szarek, I. Bengtsson, K.Ż, 2006
Quantum maps

Quantum operation: linear, completely positive trace preserving map

\[ \rho' = \Phi(\rho) = \text{Tr}_E[U(\rho \otimes \omega_E)U^\dagger]. \]

where \( \omega_E \) is an initial state of the environment while \( UU^\dagger = 1 \).

Enviromental form

Kraus form

\[ \rho' = \Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \]

where the Kraus operators satisfy \( \sum_i A_i^\dagger A_i = 1 \).
A model discrete quantum dynamics

a) unitary dynamics (rotation), $\rho' = U \rho U^\dagger$

b) decoherence (contraction), $\rho'' = \sum_i^k A_i \rho' A_i^\dagger$

Two qubit model - $N = 2 \times 2 = 4$

a) free evolution: $U = \exp(itH)$ where $H = \sigma_x \otimes \sigma_y$

(non-local unitary dynamics !)

variant b1) bistochastic channel: $\Phi(1/N) = 1/N$, 
One–qubit Pauli channel: $k = 4$, $A_1 = \sqrt{1-\epsilon} 1 \otimes 1$, $A_2 = \sqrt{\epsilon/3} 1 \otimes \sigma_x$, $A_3 = \sqrt{\epsilon/3} 1 \otimes \sigma_y$, $A_4 = \sqrt{\epsilon/3} 1 \otimes \sigma_z$.

variant b2) non bistochastic channel:
One qubit amplitude damping channel, (decaying channel), $k = 2$, 
where $A_1 = 1 \otimes B_1$ and $A_2 = 1 \otimes B_2$

with $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}$
Entanglement of formation $E$ as a function of time $t_n$

for some initially pure states of a two–qubit system.
Entanglement of formation $E$ as a function of time $t_n$

for some initially pure states of a two–qubit system.

K. Ž, P. Horodecki, M. Horodecki, R. Horodecki, PRA 2002

the name coined by Yau and Eberly, who independently reported this effect in 2003.
The effects of
a) unitary evolution (rotation of the body of mixed states) induces oscillations of entanglement
b) non–unitary evolution (decoherence): an increase of the degree of mixing (von Neumann entropy) sends the trajectory into the separable central region of the set of mixed states.
Concluding Remarks

- **Geometric approach** is useful to study quantum entanglement and its dynamics.
- In particular, geometric analysis allows one to explain the effects of entanglement revival and entanglement sudden death.

**Open problems**

- **two qubits**: What is the exact value of the HS volume of the set of the separable states?
- **three qubits** - pure states: What is the topology of the set of pure states locally equivalent to
  - a) GHZ state, \( |\psi_{\text{GHZ}}\rangle := \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \),
  - b) W–state, \( |\psi_{\text{W}}\rangle := \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) \)?
- **general case**: Describe the structure of the set of entangled states. Classify all possible schemes of dynamics of quantum entanglement.