Entanglement of Random Quantum States

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in collaboration with

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QMath 11, Hradec Kralove, September 6-10, 2010
Many thanks for an invitation to Hradec Kralove,

the city we have a special attitude:

Are there any poems or songs related to Hradec Kralove (in any language other than Czech) ??
August 1968 - 'Guests' from Poland in Hradec Kralove
August 1968 - Polish troops entering **Czechoslovakia**
Fall 1968 - city murals in Poland
(against invasion of Polish troops to the Czech Republic)
Rocznica (1968)

Ciężkie duszne iupalne
Są noce sierpniowe,
Śpią już ludzie, śpi już miasto,
Śpi **Hradec Kralowe**.

Stał mężczyzna w progu domu
I oczom nie wierzył,
**W Hradcu Kralovem, w Hradcu Kalovem**
Dudni krok żołnierzzy.

A kobieta przerażona
Patrzy zza firanki
**W Hradcu Kralovem, w Hradcu Kralovem**
Dudnią polskie tanki.

Była wolność i swoboda,
Była demokracja,
**W Hradcu Kralovem, w Hradcu Kralovem**
Polska okupacja.

Nie z twojego rozkazu
Grzmią buty wojskowe,
Lecz co ty zrobiłeś

**Dla Hradec Kralove**?

Jacek Tarkowski
Przyjaciel nikt nie będzie mi wybierał
Przyjaciel nikt nie będzie mi wybierał
Wrogów poszukam sobie sam.
...
Przebacz mi smutna Bratysławo
**Hradcu Kralowy**, złota Prago
Za śmierć jaskółki tamtej wiosny
I polskie tanki nad Wełtawą

**Andrzej Garczarek  (1981)**

---

*Sad Bratislava, **Hradec Kralove**, Golden Prague,*
*Forgive me the death of the swallow of the spring 68*
*And for Polish tanks at the Vltava*
Another Polish song concluded:

You should be ashamed
of Polish tanks in Hradec Kralove

Officially 'Polish Army was invited to Hradec Kralove'

However, up till now nobody found the person, who issued this invitation...
Letter of Invitation

Dear Professor Zyczkowski,

The programme committee of the conference “QMath11 – Mathematical Results in Quantum Physics” has suggested your name as a plenary speaker. Consequently, I have the pleasure to invite you to attend in this capacity.

The conference is to be held on September 6-10, 2010 in Hradec Králové, Czech Republic. It is the 11th conference in the QMath series, which takes place every five years.

Hradec Králové, 2nd June 2010
With the best regards,

Prof. Petr Šeba,
QMath11 local organizing committee chairman

Department of physics
University of Hradec Králové, Faculty of Education
Rokitanského 62
500 03 Hradec Králové
A friendly Polish visit to **Hradec Kralove** in 2010:

**without tanks,**

**but**

with **Random Matrices**

\[ G_{\text{rand}} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{23} \end{bmatrix} \]
Reduction of random pure states

1) Consider an ensemble of random pure states $|\psi\rangle$ of a composite system distributed according to a given measure $\mu$.

2) Perform partial trace over a chosen subsystem $B$ to get a random mixed state

$$\rho := \text{Tr}_B |\psi\rangle\langle\psi|$$

Depending on the structure of the composite system, the initial measure $\mu$ in the space of the pure states and the choice of the subsystem $B$, over which the averaging is performed one obtains different ensembles of random mixed states.
Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Space of normalized complex pure states for an arbitrary $N$:

Since $\langle \psi | \psi \rangle = 1$ a **normalized** state belongs to the **sphere** $S^{2N-1}$.

Two states equal up to a phase are identified, $|\psi\rangle \sim e^{i\alpha} |\psi\rangle$, so the set of states is equivalent to the **complex projective space** $\mathbb{C}P^{N-1}$ of $2N - 2$ real dimensions.

$N = 2$: For **qubit** = quantum bit the word **geometry** can be treated literally!

$|\psi\rangle = \cos \frac{\vartheta}{2} |1\rangle + e^{i\phi} \sin \frac{\vartheta}{2} |0\rangle$

$\mathbb{C}P^1 = $ **Bloch sphere** of $N = 2$ pure states
Random Pure states in $\mathcal{H}_N$

'Quantum chaotic' dynamics (pseudo-random evolution)
described by a random unitary matrix $U$ acting on a pure state produces (almost surely) a 'generic pure state' $|\psi\rangle = U|\phi_0\rangle$.

- Formally one defines an (unique) Fubini–Study measure $\mu$ on complex projective spaces which is unitarily invariant: for any (measurable) set $A$ of states one requires $\mu(A) = \mu(U(A))$.

- This measure covers the entire space $\mathbb{C}P^{N-1}$ uniformly, and for $N = 2$ it is just equivalent to the uniform, Lebesgue measure on the sphere $S^2$.

How to obtain numerically a random pure state $|\psi\rangle$?

a) Take a column (a row) of a random unitary $U$ so that $|\psi\rangle = U|i\rangle$.

b) generate $N$ independent complex random numbers $z_i$ according to the normal distribution. Write $|\psi\rangle = \sum_{i=1}^{N} c_i |i\rangle$ where the expansion coefficients read $c_i = z_i / \sqrt{\sum_i |z_i|^2}$. 
One quantum state fixed, one random...

Fix an arbitrary state $|\psi_1\rangle$. Generate randomly the other state $|\psi_2\rangle$.

- What is the average angle $\chi$ between these states?
- What is the distribution $P(\chi)$ of the angle $\chi := \arccos |\langle \psi_1 | \psi_2 \rangle|$?
One quantum state fixed, one random...

**Fix an arbitrary state** $|\psi_1\rangle$. Generate randomly the other state $|\psi_2\rangle$.

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Measure concentration phenomenon

'Fat hyper-equator' of the sphere $S^N$ in $\mathbb{R}^{N+1}$...

It is a consequence of the Jacobian factor for expressing the volume element of the $N$– sphere. Let $z = \cos\vartheta_1$, so that

$$J \sim (\sin\vartheta_1)^{N-1} J_2(\vartheta_2, \ldots, \vartheta_N)$$

Hence the typical angle $\chi$ is 'close' to $\pi/2$ and two 'typical random states' are orthogonal and the distribution $P(\chi)$ is 'close' to $\delta(\chi - \pi/2)$.

How close?
Levy’s Lemma (on higher dimensional spheres)

Let $f : S^N \to \mathbb{R}$ be a Lipschitz function, with the constant $\eta$ and the mean value $\langle f \rangle = \int_{S^N} f(x) d\mu(x)$.

Pick a point $x \in S^N$ at random from the sphere. For large $N$ it is then unlikely to get a value of $f$ much different then the average:

$$P\left( |f(x) - \langle f \rangle| > \alpha \right) \leq 2 \exp\left( -\frac{(N+1)\alpha^2}{9\pi^3\eta^2} \right)$$

Simple application: the distance from the 'equator'.

Take $f(x_1, \ldots, x_{N+1}) = x_1$. Then Levy’s Lemma says that the probability of finding a random point of $S^N$ outside a band along the equator of width $2\alpha$ converges exponentially to zero as $2 \exp[-C(N+1)\alpha^2]$.

As $N \gg 1$ then every equator of $S^N$ is 'FAT'.

a walk through Hradec Kralove
Composed systems & entangled states

**Bi-partite systems:** \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \)

- **Separable pure states:** \( |\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \)
- **Entangled pure states:** all states **not** of the above product form.

**Two-qubit system:** \( d = 2 \times 2 = 4 \)

Maximally entangled **Bell state** \( |\varphi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \)

**Entanglement measures**

For any pure state \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) define its partial trace \( \sigma = \text{Tr}_B |\psi\rangle\langle \psi| \).

**Definition:** **Entanglement entropy** of \( |\psi\rangle \) is equal to von Neumann entropy of the partial trace

\[
E(|\psi\rangle) := -\text{Tr} \sigma \ln \sigma
\]

The more mixed partial trace, the more entangled initial pure state...
Entanglement of two real qubits

Entanglement entropy at the tetrahedron of $d = 4$ real pure states
More on this is can be found in

I. Bengtsson and K. Życzkowski, *Geometry of Quantum States*
(Cambridge, 2006, 2008)
Generic pure states of a bi-partite system

'Two quNits' = \( N \times N \) quantum system

The space \( \mathbb{C}P^{N^2-1} \) of all states in \( \mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N \) has \( d_{\text{tot}} = N^2 - 2 \) dimensions.

The subspace of separable (product) states \( \mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1} \) has only \( d_{\text{sep}} = 2(N-2) \) dimensions. For large \( N \) we observe that \( d_{\text{sep}} \sim 2N \ll d_{\text{tot}} \sim N^2 \) so the separable states form a set of measure zero in the space of all states.

Thus a 'typical' random state is entangled!

How much entangled?

Mean entropy of the reduced density matrix \( \rho \)

Let us call \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). Take any pure state \( |\psi\rangle \in \mathcal{H} \) and define its partial trace \( \rho := \text{Tr}_B |\psi\rangle\langle\psi| = \text{Tr}_A |\psi\rangle\langle\psi| \).

The von Neummann entropy \( S \) of the reduced mixed state \( \rho \) is a measure of entanglement of the initially pure bi-partite state \( |\psi\rangle \).
Average entanglement entropy for a bipartite system

**N × N system**

\[ \langle S(\psi) \rangle_\psi \approx \ln N - \frac{1}{2} + \mathcal{O}\left(\frac{\ln N}{N}\right) \]


valid for random states in \( \mathcal{H}_N \otimes \mathcal{H}_K \) with \( K \geq N \)

\[ \langle S(\psi) \rangle_\psi = \Psi(NK + 1) - \Psi(K + 1) - \frac{N - 1}{2K} \approx \ln N - \frac{N}{2K} \]

**N × K system: probability measure**

Let \( \lambda = \{\lambda_1, \ldots, \lambda_N\} \) denote the spectrum of the reduced matrix \( \rho := \text{Tr}_B |\psi\rangle\langle\psi| \). If \( |\psi\rangle \) is taken uniformly on \( \mathcal{H}_N \otimes \mathcal{H}_K \) then

\[ P_{N,K}(\lambda) = C_{N,K} \delta(1 - \sum_i \lambda_i) \prod_i \lambda_i^{K-N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \]

normalization constants \( C_{N,K} \) derived in Sommers, Życzkowski (2001)
Concentration of entropy of the partial trace

Consider an $N \times K$ system with $K \geq N$

The maximal entropy (achieved for $\rho^* = \mathbb{1}_N/N$) is equal to $S_{\text{max}} := \ln N$.

Since the mean entropy, $\langle S \rangle_\psi \approx S_{\text{max}} - N/2K$, is close to the maximal value a concentration effect has to occur...

Levy’s lemma and concentration of entanglement

Consider the sphere $S^{2NK-1}$ which represents pure states of a $N \times K$ system with $K \geq N \geq 3$. Use Levy’s lemma with $f = S(\rho)$. It implies

$$P\left(S(\text{Tr}_B|\psi\rangle\langle\psi|) < \ln N - N/2K - \alpha\right) \leq \exp\left(-\frac{(NK - 1)}{8(\pi \ln N)^2} \alpha^2\right)$$


Thus the reduced density matrix $\rho$ is close to the maximally mixed state $\rho^* = \mathbb{1}_N/N$, while the initial random pure state is close to a maximally entangled state $|\psi^+\rangle$ with entropy $S_{\text{max}} = \ln N$. 
Composed bi–partite systems on $\mathcal{H}_A \otimes \mathcal{H}_B$

Partial trace over one subsystem produces mixed state

Consider an ensemble of random pure states $|\psi\rangle$ distributed according to a given measure $\mu$. Define a reduced mixed state $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$. 

Ensembles obtained by partial trace: a) induced measure

i) natural measure on the space of pure states obtained by acting on a fixed state $|0,0\rangle$ with a global random unitary $U_{AB}$ of size $KN$.

ii) partial trace over the $K$ dimensional subsystem $B$ leads to the induced measure $P_{N,K}(\lambda)$ in the space of mixed states of size $N$. Integrating out all eigenvalues but $\lambda_1$ one arrives (for large $N$) at the Marchenko–Pastur distribution $P_c(x = N\lambda_1)$ with the parameter $c = K/N$. 

\[ K \cdot Z (I F U J / C F T P A N ) \]
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Spectral properties of random matrices

Non-hermitian matrix $G$ of size $N$ of the Ginibre ensemble

Under normalization $\text{Tr} GG^\dagger = N$ the spectrum of $G$ fills uniformly (for large $N$!) the unit disk.

The so-called circular law!

Hermitian, positive matrix $\rho = GG^\dagger$ of the Wishart ensemble

Let $x = N\lambda_i$, where $\{\lambda_i\}$ denotes the spectrum of $\rho$. As $\text{Tr}\rho = 1$ so $\langle x \rangle = 1$. Distribution of the spectrum $P(x)$ is asymptotically given by the Marchenko–Pastur law

$$\pi^{(1)}(x) = P_{\text{MP}}(x) = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1} \quad \text{for} \quad x \in [0, 4]$$
'Biased' ensembles of bi–partite states

Consider a superposition of a given **bi-partite state** $|\phi_{AB}\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$ with the same state transformed by a random **local unitary** $U_A$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |\phi_{AB}\rangle + (U_A \otimes \mathbb{1}_N)|\phi_{AB}\rangle \right)$$

Is the outcome superposition state $|\psi\rangle$ (on average) **more entangled** than the initial $|\phi_{AB}\rangle$?

What reduced states are (on average) **more mixed**:

$$\rho = \text{Tr} |\phi_{AB}\rangle\langle \phi_{AB}| \quad \text{or} \quad \rho' = \text{Tr} |\psi\rangle\langle \psi|$$
b) Arcsine ensemble

i) Consider a superposition of \textbf{two maximally entangled} states on \( \mathcal{H}_N \otimes \mathcal{H}_N \)

\[
|\phi\rangle = |\psi_{AB}^+\rangle + (U_A \otimes 1_N)|\psi_{AB}^+\rangle,
\]

where \( |\psi_{AB}^+\rangle = (1/\sqrt{N}) \sum_{i=1}^N |i, i\rangle \), while \( U_A \in U(N) \) is a \textbf{Haar random unitary matrix} with phases \( \alpha_i \).

ii) The reduced state \( \rho_A = \text{Tr}_B |\phi\rangle \langle \phi| \)

has the spectrum \( \lambda_i = (1 + \cos \alpha_i)/N \) for \( i = 1, \ldots, N \). Thus for large \( N \) the spectral density has the form of the \textbf{arcsine distribution},

\[
P_{\text{arc}}(x) = \frac{1}{\pi \sqrt{x(2-x)}}
\]

with support \( x \in [0, 2] \), where \( x = N\lambda \).
c) Bures ensemble

i) Consider a superposition of two pure states: a random state $|\psi_1\rangle$ and the same state transformed by a **local unitary** $V_A$,

$$|\phi\rangle := (1 \otimes 1 + V_A \otimes 1)|\psi_1\rangle,$$

where $|\psi_1\rangle = U_{AB}|0,0\rangle$ while $V_A \in U(N)$ and $U_{AB} \in U(N^2)$ are **Haar random unitary matrices**.

![Diagram](image)

ii) The reduced state $\rho_B = \frac{(1+V_A)GG^\dagger(1+V_A^\dagger)}{\text{Tr}[(1+V_A)GG^\dagger(1+V_A^\dagger)\]}$ is distributed according to the **Bures measure**, $P_B(\lambda_1, \ldots, \lambda_N) = C_N^B \prod_i \lambda_i^{-1/2} \prod_{i<j}^1 \prod_{N} \frac{(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}$ (Osipov, Sommers, Žyczkowski, 2010) characterized by the **Bures distribution**,

$$P_B(x) = \frac{1}{4\pi\sqrt{3}} \left[ \left( \frac{a}{x} + \sqrt{\left( \frac{a}{x} \right)^2 - 1} \right)^{2/3} - \left( \frac{a}{x} - \sqrt{\left( \frac{a}{x} \right)^2 - 1} \right)^{2/3} \right]$$

where $a = 3\sqrt{3}$. Square matrix $G$ of size $N$ from the **Ginibre ensemble** is obtained from the first column of $U_{AB}$ of size $N^2$ which acts on $|0,0\rangle$. 
a) **Four-partite system & \(\pi^{(2)}\) distribution**

Take a four-partite product state,

\[ |\psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D =: |0,0,0,0\rangle \in \mathcal{H}_N^{\otimes 4}.\]

i) Apply two random unitary matrices \(U_{AB}\) and \(U_{CD}\) of size \(N^2\),

\[ |\psi\rangle = U_{AB} \otimes U_{CD} |\psi_0\rangle = \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} G_{ij} E_{kl} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D \]

ii) Consider projector \(P := 1_A \otimes |\Psi_{BC}^+\rangle\langle\Psi_{BC}^+| \otimes 1_D\) on the maximally entangled state,

\[ |\Psi_{BC}^+\rangle = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} |\mu\rangle_B \otimes |\mu\rangle_C \]

The spectrum of the iii) reduced state \(\rho_A = \frac{\text{Tr}_D |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G_{EE}^\dagger G^\dagger}{\text{Tr} G_{EE}^\dagger G^\dagger}\) consists of squared singular values of the product \(GE\) of two independent Ginibre matrices, so the spectral density is described by the **Fuss-Catalan distribution** \(\pi^{(2)}(x)\).
Take a 2s-partite product state,

$$|\psi_0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s} \in \mathcal{H}_N \otimes 2s.$$  

i) Apply $s$ random unitary matrices $U_{1,2}, U_{3,4}, \ldots, U_{2s-1,2s}$ of size $N^2$ each,

$$|\psi\rangle U_{1,2} \otimes \cdots \otimes U_{2s-1,2s} |0, \ldots, 0\rangle = \sum_{i_1, \ldots, i_{2s}} (G_1)_{i_1,i_2} \cdots (G_s)_{i_{2s-1},i_{2s}} |i_1, \ldots, i_{2s}\rangle$$  

ii) Project onto the product of $(s - 1)$ maximally entangled states,

$$P_s := 1_1 \otimes |\Psi_{2,3}^+\rangle \langle \Psi_{2,3}^+| \otimes \cdots \otimes |\Psi_{2s-2,2s-1}^+\rangle \langle \Psi_{2s-2,2s-1}^+| \otimes 1_{2s}$$

The spectrum of the iii) reduced state

$$\rho_A = \frac{\text{Tr}_{2s} |\phi\rangle \langle \phi|}{\langle \phi| \phi\rangle} = \frac{G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger}{\text{Tr} [G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger]}$$

consists of squared singular values of the product $G_1 \cdots G_s$ of $s$ independent Ginibre matrices, so the spectral density is described by the Fuss-Catalan distribution $\pi^{(s)}(X)$. 

### Diagram

![Diagram of a 2s-partite system with unitary matrices and projectors](http://example.com/diagram.png)
Fuss-Catalan distribution $\pi^{(s)}$

defined for an integer number $s$ is characterized by its moments

$$\int x^p \pi^{(s)}(x) dx = \frac{1}{sp+1} \left( \frac{sp+p}{p} \right) =: FC_p^{(s)}$$
equal to the general\textit{ized} Fuss-Catalan numbers.

The density $\pi^{(s)}$ is analytic on the support $[0, (s+1)^{s+1}/s^s]$, while for $x \to 0$ it behaves as $1/(\pi x^{s/(s+1)})$.

Density $\pi^{(s)}$ can be expressed as a sum of $s$ hypergeometric functions

$$s F_{s-1}(a_1, \ldots, a_s; b_1, \ldots, b_{s-1}; \alpha x)$$

(Penson, Życzkowski, 2010)
Spectral properties of the ensembles analyzed

Spectral density $P(x)$ of the rescaled eigenvalue $x = N\lambda$

<table>
<thead>
<tr>
<th>matrix $W$</th>
<th>$P(x)$</th>
<th>$x \rightarrow 0$</th>
<th>support</th>
<th>mean entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{1}$</td>
<td>$\pi^{(0)}$</td>
<td>$-$</td>
<td>${1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\mathbb{1} + U$</td>
<td>arcsine</td>
<td>$x^{-1/2}$</td>
<td>$[0, 2]$</td>
<td>$\ln 2 - 1 \approx -0.307$</td>
</tr>
<tr>
<td>$G$</td>
<td>$\pi^{(1)}$</td>
<td>$x^{-1/2}$</td>
<td>$[0, 4]$</td>
<td>$-1/2 = -0.5$</td>
</tr>
<tr>
<td>$(\mathbb{1} + U)G$</td>
<td>Bures</td>
<td>$x^{-2/3}$</td>
<td>$[0, 3\sqrt{3}]$</td>
<td>$-\ln 2 \approx -0.693$</td>
</tr>
<tr>
<td>$G_1 G_2$</td>
<td>F–C</td>
<td>$x^{-2/3}$</td>
<td>$[0, 6^{3/4}]$</td>
<td>$-5/6 \approx -0.833$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$G_1 \cdots G_s$</td>
<td>F–C</td>
<td>$x^{-s/(s+1)}$</td>
<td>$[0, b_s]$</td>
<td>$- \sum_{j=2}^{s+1} \frac{1}{j}$</td>
</tr>
</tbody>
</table>

Table: Ensembles of random mixed states obtained as normalized Wishart matrices, $\rho = WW^\dagger / \text{Tr} WW^\dagger$. Here $b_s = (s + 1)^{s+1}/s^s$ and the mean entropy $\langle S \rangle = - \int x \ln x P(x) dx$. 
**Generalized ensemble of random states**

Let

\[
W_{k,s} := \left( U_1 + U_2 + \cdots + U_k \right) G_1 \cdots G_s
\]

where \( U_i \) are independent **Haar random unitary** matrices, while \( G_i \) are independent random **Ginibre matrices**.

Define generalized ensemble of normalized random density matrices

\[
\rho_{k,s} := \frac{W_{k,s} W_{k,s}^\dagger}{\text{Tr}(W_{k,s} W_{k,s}^\dagger)}
\]

**Special cases:**

- \( s = 0, \ k = 1 \) \quad \Rightarrow \quad \text{maximally mixed state}
- \( s = 0, \ k = 2 \) \quad \Rightarrow \quad \text{arcsine ensemble}
- \( s = 1, \ k = 1 \) \quad \Rightarrow \quad \text{Hilbert-Schmidt ensemble}
- \( s = 1, \ k = 2 \) \quad \Rightarrow \quad \text{Bures ensemble}
- \( s = s, \ k = 1 \) \quad \Rightarrow \quad s – \text{Fuss Catalan ensemble}
Multi–partite systems: graphs

Graph random states

Consider a graph \( \Gamma \) consisting of \( m \) edges \( B_1, \ldots B_m \) and \( k \) vertices \( V_1, \ldots V_k \). It represents a composite quantum system consisting of \( 2m \) sub–systems described in the Hilbert space with \( 2m \)–fold tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m} \) of dimension \( N^{2m} \).

Each edge represents the maximally entangled state \( |\Phi^+\rangle \) in both subspaces, while each vertex represents a random unitary matrix \( U \) (Haar measure = ’generic’ Hamiltonian), coupling connected systems.

A simple example: three vertices & two edges

We define a random state

\[
|\psi\rangle = (U_1 \otimes U_{23} \otimes U_4) |\Phi^{+}_{12}\rangle \otimes |\Phi^{+}_{34}\rangle
\]

where \( |\Phi^{+}_{kj}\rangle \) denotes the maximally entangled state in subspaces \( k, j \).
Consider an ensemble of random pure states $|\psi\rangle$ corresponding to a given graph $\Gamma$. Select a fixed subset $T$ of subspaces and define a (random) mixed state $\rho(T) = \text{Tr}_{T} |\psi\rangle\langle\psi|$. 

**Tasks**

- Determine the spectral properties of the ensemble of mixed states $\rho(T)$ associated with the graph $\Gamma$.
- Find the mean entropy $\langle S(\rho) \rangle_{\psi}$ of the reduced state $\rho$ averaged over the ensemble of graph random pure states $|\psi\rangle_{\Gamma,T}$.

**Examples of partial trace for the graph $\Gamma$**

The partial trace is taken over all the subspaces $T$ represented by open symbols.
Graphs and random multi–partite systems

Partial trace over certain subspaces

For ensembles of random states associated with certain graphs $\Gamma$ and selected subspaces $T$ – cross ($\times$) – over which the partial trace takes place,

$$
\text{one can compute moments of the traces } \mu_q := \langle \text{Tr} \rho^q \rangle_\psi
$$

and then obtain bounds for the average entropy

$$
\langle S \rangle = \langle -\text{Tr} \rho \ln \rho \rangle_\psi.
$$

Collins, Nechita, Žyczkowski, J. Phys. A,
Example 1: 2 bonds, 4 subsystems and one bi-partite interaction $U_0$

a) $\pi^{(0)}$ – maximaly mixed state $\rho = \frac{1}{N} \mathbb{1}$ with entropy $S(\rho) = \ln N$

\[ V_1 \quad \quad V_3 \quad \quad V_2 \]

or

\[ V_1 \quad \quad V_3 \quad \quad V_2 \]

b) $\pi^{(1)}$ random mixed state generated according to the induced measure

\[ V_1 \quad \quad V_3 \quad \quad V_2 \]

with entropy $S(\rho) \approx \ln N - 1/2$

Let $|\psi\rangle = \sum_i \sum_j G_{ij} |i\rangle \otimes |j\rangle$ be a random pure state.

Then $G$ is a random matrix of Ginibre ensemble consisting of independent complex Gaussian entries normalized as $|G|^2 = \text{Tr} GG^\dagger = 1$.

The distribution of eigenvalues of a non–hermitian matrix $G$ is given by the Girko circular law, while positive Wishart matrices $\rho = \text{Tr}_B |\psi\rangle \langle \psi| = GG^\dagger$ are described by Marchenko-Pastur law $\pi^{(1)}$. 
Example 2: 4 bonds, 8 subsystems and four bi-partite interactions $V_i$

c) $\pi^{(2)}$ random mixed state generated by the 4–cycle graph

After partial trace over crossed subsystems the random mixed state has the structure

$$\rho = \alpha G_2 G_1 G_1^\dagger G_2^\dagger,$$

where $G_1$ and $G_2$ are independent Ginibre matrices and $\alpha = 1/\text{Tr} G_2 G_1 G_1^\dagger G_2^\dagger$.

Mixed states with spectrum given by the Fuss-Catalan distribution $\pi^{(2)}(x)$ characterized by mean entropy

$$S(\rho) \approx \ln N - 5/6$$

$$P_{\text{MP}}(x) = \pi^{(1)}(x) \text{ and } \pi^{(2)}(x).$$
Multi-partite systems: a lattice $L$

Partition of the lattice into two disjoint sets, $L = A \cup \bar{A}$

Consider lattice (graph), in which each **vertex** denotes a spin

(*different meaning than before!*)

and each **edge** represents an interaction defined by a local Hamiltonian $H$.

Let $A$ denotes a distinguished set of vertices while $\partial A$ represents **spins**

belonging to its **area**, i.e. these spins for which some edges are cut away.
Consider an eigenstate $|\psi\rangle$ of the Hamiltonian $H$, define set of spins $A$ and take the partial trace of the pure state over all spins belonging to the complementary set $\bar{A}$.

- **Von Neumann entropy** of the resulting mixed state $\rho := \text{Tr}_{\bar{A}}|\psi\rangle\langle \psi|$ is proportional to the **area** $\partial A$ of the distinguished subset $A$. Hence **entanglement** of the state $|\psi\rangle$ with respect to the partition $A \cup \bar{A}$ behaves as the **area** $\partial A$.

Theorem. Consider a graph $\Gamma$ and its partition into two sets $A$ and $\bar{A}$. Let $|\psi\rangle$ be a random graph pure state and $\rho := \text{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$. Then the mean entropy of $\rho$ (entanglement entropy of $|\psi\rangle$) is proportional to the number $M$ of bonds cut ('area' of $A$),

$$\langle S(\rho) \rangle_{\psi} = M \ln N.$$ 

Example: graph with 10 bonds, $M = 5$ of them cut

The area law $S(\rho) = 5 \ln N$ is universal as it does not depend on the choice of Hamiltonians describing the interaction in the vertices.

Only the topology of the interaction matters!
Concluding remarks

- There exists a natural, unitarily invariant measure in the space $\mathbb{C}P^{N-1}$ of pure states of a finite size $N$. A quantized chaotic evolution sends an initial state $|i\rangle$ into a 'typical' state $|\psi\rangle$.
- A generic pure state of a bi-partite quantum system is strongly entangled, so its partial trace is strongly mixed!
- 'Biased' ensembles of random pure states + partial trace allow one to generate random states according to various measures, including (Arcsine, Hilbert-Schmidt, Bures, s–Fuss-Catalan) ensembles.
- With any graph one can associate an ensemble of random pure states. Selecting a set $A$ of subsystems we define an ensemble of mixed states $\rho$ by performing the partial trace over them. Statistics of the spectra of $\rho$ is described by delta distribution $h_0(x) = \delta(x - 1)$, Marchenko–Pastur distribution $h_1(x)$ or Fuss–Catalan distributions $h_s(x)$, with $s \geq 2$, for which mean entropies are known.
- Universal Entanglement Area law: For any graph $\Gamma$ and its partition $A$ and $\bar{A}$ the mean entanglement entropy of the random pure state $|\psi\rangle$ depends on the area $\partial A$ (the number of bonds cut).