Joint Numerical Range
of three hermitian matrices of size three

(3D Numerical Range)

Karol Życzkowski

in collaboration with

Konrad Szymański (Cracow) & Stephan Weis (Campinas),

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Dedicated to memory of Eugene Gutkin (1946-2013)
Quantum physics & matrices

Quantum states = statics
To describe objects of the micro world quantum physics uses quantum states: vectors from a $N$-dimensional Hilbert space. In Dirac notation:

a) ket: $|\psi\rangle = (z_1, z_2, \ldots, z_N)^* \in \mathcal{H}_N$,
b) bra: $\langle \psi | = (z_1, z_2, \ldots, z_N) \in \mathcal{H}_N^*$,
c) bra-ket $\langle \psi | \phi \rangle \in \mathbb{C} = (\psi, \phi)$ is a scalar product,
d) ket-bra $|\psi\rangle \langle \phi |$ is an operator = a matrix of order $N$.

Mixed quantum states

$\rho = \sum_i a_i |\psi_i\rangle \langle \psi_i |$ where $a_i \geq 0$ and $\sum_i a_i = 1$ (matrix of order $N$)

= normalized convexed combination of projection operators $P_\psi = |\psi\rangle \langle \psi |$

Discrete Dynamics: Quantum maps

Quantum map $\Phi : \rho' = \Phi(\rho)$ – a linear operation acting on matrices of order $N$. A map $\Phi$ can be represented by a matrix of order $N^2$. 
Mixed quantum states = density matrices

Set $\mathcal{M}_N$ of all mixed states of size $N$

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^*, \rho \geq 0, \text{Tr}\rho = 1\}$$

Example: $N = 2$, One-qubit states:

$$\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$$\rho = \sum_i a_i |\psi_i\rangle \langle \psi_i|$$ where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

The set $\mathcal{M}_3$ of all $N = 3$ mixed states

$\mathcal{M}_3$ is 8 dimensional convex set in $\mathbb{R}^8$

with 4 dimensional simply connected subset $\mathbb{C}P^2$ of pure (extremal) states, which belongs to 7 dimensional boundary $\partial \mathcal{M}_3$. 
The set $\mathcal{M}_N$ of **quantum mixed states** for $N \geq 3$

A constructive approach:

Analysis of its structure with aid of notions of operator theory like **Numerical Range**

The same tools are useful to investigate the structure of the subsets of $\mathcal{M}_N$, namely sets of

a) **separable** states

and

b) **maximally entangled** states.
Definition

For any operator $A$ acting on $\mathcal{H}_N$ one defines its **NUMERICAL RANGE** (Wertevorrat) as a subset of the complex plane defined by:

$$\Lambda(A) = \{ \langle x|A|x\rangle : |x\rangle \in \mathcal{H}_N, \langle x|x\rangle = 1 \}.$$  \hspace{1cm} (1)

Hermitian case

For any hermitian operator $A = A^\dagger$ with spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ its **numerical range** forms an interval: the set of all possible expectation values of the observable $A$ among arbitrary pure states, $\Lambda(A) = [\lambda_1, \lambda_N]$. 

\[ \begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\hline
\end{array} \]

$N=4$
Numerical range and its properties

Compactness

\( \Lambda(A) \) is a **compact** subset of \( \mathbb{C} \).

Convexity: Hausdorff-Toeplitz theorem

- \( \Lambda(A) \) is a **convex** subset of \( \mathbb{C} \).

Example

Numerical range for random matrices of order \( N = 6 \)

- a) normal,
- b) generic (non-normal)
Numerical range & quantum states:

Set of pure states of a finite size $N$:

Let $\Omega_N = \mathbb{C}P^{N-1}$ denotes the set of all pure states in $\mathcal{H}_N$.

**Example:** for $N = 2$ one obtains **Bloch sphere**, $\Omega_2 = \mathbb{C}P^1$.

Probability distributions on complex plane supported in $\Lambda(A)$

- **Fix** an arbitrary operator $A$ of size $N$,
- **Generate** random pure states $|\psi\rangle$ of size $N$
  (with respect to unitary invariant, **Fubini–Study** measure) and analyze their contribution $\langle \psi | A | \psi \rangle$ to the numerical range on the complex plane....

**Probability distribution:** (**'numerical shadow'** – **'image'** of a matrix)

$$P_A(z) := \int_{\Omega_N} d\psi \; \delta(z - \langle \psi | A | \psi \rangle)$$

is supported on $\Lambda(A)$.

How does it look like?
Normal case: a ’shadow’ of the classical simplex...

Normal matrices of size $N = 2$ with spectrum $\{\lambda_1, \lambda_2\}$

Distribution $P_A(z)$ covers **uniformly** the interval $[\lambda_1, \lambda_2]$ on the complex plane,

Examples for diagonal matrices $A$ of size **two** and **three**

![Density plot of numerical range of operator $[1, 0; 0, i]$](image1)
![Density plot of numerical range of operator $\text{diag}(1, i, -i)$](image2)

Normal matrices of order $N = 3$ with spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$

Distribution $P_A(z)$ covers **uniformly** the triangle $\Delta(\lambda_1, \lambda_2, \lambda_3)$ on the complex plane.
Normal case: a 'shadow' of the classical simplex...

Normal matrices of size $N = 4$ with spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$

Distribution $P_A(z)$ covers the quadrangle $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with the density determined by projection of the regular tetrahedron (3–simplex) on the complex plane.

Examples for diagonal matrices $A$ of size four. Observe shadow of the edges of the tetrahedron = the set of all classical mixed states for $N = 4$. 
Numerical range for matrices of order \( N = 2 \) with spectrum \( \{\lambda_1, \lambda_2\} \).

a) **normal** matrix \( A \Rightarrow \Lambda(A) = \text{closed interval} \ [\lambda_1, \lambda_2] \)

b) **not normal** matrix \( A \Rightarrow \Lambda(A) = \text{elliptical disk} \) with \( \lambda_1, \lambda_2 \) as focal points and minor axis, \( d = \sqrt{\text{Tr} AA^* - |\lambda_1|^2 - |\lambda_2|^2} \)
(Murnaghan, 1932; Li, 1996).

**Example:** The **Jordan matrix** \( J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \).

Its numerical range forms a circular disk, \( \Lambda(J) = D(0, r = 1/2) \).

The set of \( N = 2 \) pure quantum states
A projection of the **Bloch sphere** \( S^2 = \mathbb{C}P^1 \) onto a plane forms an **ellipse**, (which could be degenerated to an interval).
Numerical shadow for $N = 2$

Non–normal matrices of size $N = 2$

Distribution $P_A(z)$ covers entire (elliptical) disk on the complex plane with non-uniform (!) density determined by projection of (empty!) Bloch sphere, $S^2 = \mathbb{C}P^1$ on the complex plane.

Examples for non-normal matrices $A$ of size two. Note the 'shadow' of the projective space $\mathbb{C}P^1 = S^2 =$ set of quantum pure states for $N = 2$. 
Shadows of three dimensional objects...

- Light direction

- a)
- b)
- c)
Quantum States and Numerical Range/Shadow

Classical States & normal matrices

Proposition 1. Let $C_N$ denote the set of classical states of size $N$, which forms the regular simplex $\Delta_{N-1}$ in $\mathbb{R}^{N-1}$. Then the set of similar images of orthogonal projections of $C_N$ on a 2–plane is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of all normal matrices $A$ of order $N$ (such that $AA^* = A^*A$).

Quantum States & non–normal matrices

Proposition 2. Let $M_N$ denotes the set of quantum states size $N$ embedded in $\mathbb{R}^{N^2-1}$ with respect to Euclidean geometry induced by Hilbert-Schmidt distance. Then the set of similar images of orthogonal projections $M_N$ on a 2-plane is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of all matrices $A$ of order $N$. 
Normal case: a 'shadow' of the 4-D simplex…

Normal matrices of size \(N = 5\) with spectrum \((\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\)

Distribution \(P_A(z)\) covers the pentagon \(\{\exp(2\pi j/5)\}, \ j = 0, \ldots, 4\) with the density determined by projection of the 4-D simplex on the complex plane.
How a complex / real projective space looks like?

Not normal matrices of size $N = 3$

Distribution $P_A(z)$ covers the numerical range $\Lambda(A)$ on the complex plane with a non-uniform density

'shadow' of the projective space $\Omega_3 = \mathbb{C}P^2$ (left) and $\mathbb{R}P^2$ (right) for a $N = 3$ non-normal matrix $A = [0, 1, 1; 0, i, 1; 0, 0, -1]$. 
Shadows of typical matrices of size $N = 3$

belong to one of four different classes specified e.g. by the number $s$ of flat segments of the boundary, $s = 0, 1, 2, 3$. 
Numerical range for matrices of order $N = 3$.

Classification by Keeler, Rodman, Spitkovsky 1997

Numerical range $\Lambda$ of a $3 \times 3$ matrix $A$ forms:

a) $\Lambda(A)$ is a compact set of an 'ovular' shape
   (which contains three eigenvalues!) – the generic case, $s = 0$

b) a compact set with one flat part (e.g. convex hull of a cardioid), $s = 1$

c) a compact set with two flat parts
   (e.g. convex hull of an ellipse and a point outside it), $s = 2$

d) triangle of eigenvalues, $\Lambda(A) = \Delta(\lambda_1, \lambda_2, \lambda_3)$
   for any normal matrix $A$ one has $s = 3$

These four cases describe the shape of possible projections of the complex projective space $\mathbb{C}P^2$ (embedded into $\mathbb{R}^8$) onto a 2–plane.
Joint Numerical Range (JNR) of a set of $m$ operators

$$\Lambda(A_1, A_2, \ldots, A_m) = (\langle \psi | A_1 | \psi \rangle, \langle \psi | A_2 | \psi \rangle, \ldots, \langle \psi | A_m | \psi \rangle) \subset \mathbb{R}^m.$$  

For $m \geq 3$ JNR is (in general) not a convex set!

Set $m = 2$, decompose $A = A_H + iA_A$ into its Hermitian and anti-Hermitian part. Then $\Lambda(A) = \Lambda(A_H, A_A)$

Proposition 3. Take a set $\{A_1, \ldots, A_{N^2-1}\}$ of matrices of size $N$ forming an orthonormal basis in the space of Hermitian, traceless matrices. Then

- $\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ is affine isomorphic to the set $\Omega_N = \mathbb{C}P^{N-1}$ of pure quantum states of size $N$ (embedded in $\mathbb{R}^{N^2-1}$).
- The convex hull of $\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ is isomorphic to the set $\mathcal{M}_N$ of mixed quantum states of size $N$.
- $\Lambda(A_1, A_2, \ldots, A_m)$ with $m \leq N^2 - 1$ forms a projection of $\Omega_N$ into $\mathbb{R}^m$. 
Joint Numerical Range: some examples

\(N = 2:\) one qubit states

Let \(\sigma_1, \sigma_2, \sigma_3\) denote three trace-less Pauli matrices of size \(N = 2\).

Then

- \(\Lambda(\sigma_1, \sigma_2, \sigma_3) = \Omega_2 = \mathbb{C}P^1\) forms the Bloch sphere \(S^2\) of all one–qubit pure states.
- The convex hull of \(\Lambda(\sigma_1, \sigma_2, \sigma_3)\) forms the Bloch ball, \(\mathcal{M}_2 = B_3 \subset \mathbb{R}^3\) of all one–qubit mixed states.

\(N = 3:\) one qutrit states

Let \(\lambda_1, \ldots \lambda_8\) denote eight traceless Gell–Man matrices of size 3: the generators of \(SU(3)\).

Then

- \(\Lambda(\lambda_1, \ldots \lambda_8) = \Omega_3 = \mathbb{C}P^2\) forms the set of all one–qutrit pure states.
- The convex hull of \(\Lambda(\lambda_1, \ldots, \lambda_8)\) forms the set of \(N = 3\) mixed states – a convex body \(\mathcal{M}_3\) embedded in \(\mathbb{R}^8\).
Joint Numerical Range: 3D examples for $m = 3$

$N = 3$ : one qutrit

Take any triple of hermitian operators \( \{A_1, A_2, A_3\} \) of size $N = 3$.

Then joint numerical range \( \Lambda(A_1, A_2, A_3) \subset \mathbb{R}^3 \) gives a projection of the 8D set $\mathcal{M}_3$ of mixed states of a qutrit into 3D.

Examples:

Different classes of 3D JNR: their further projections into 2D belong to one of four classes of Keeler et al. – the possible shapes of the standard numerical range for $N = 3$. 
Konrad Szymański producing a 3D joint numerical range
Recall the shadows on the wall of the cave of *Plato*:

we do not understand all details of the 8D set $\mathcal{M}_3$ of quantum states of size three, but at least we can study its 2D and 3D *projections*.

How to classify possible shapes of JNR of three Hermitian matrices $A_1, A_2, A_3$ of size $N = 3$?
Geometric approach

To classify the 3D numerical ranges for each body we count:

a) the number \( s \) of flat **segments** in the boundary
b) the number \( e \) of flat **faces (ellipses)** in the boundary

eight standard cases (above) + two 'singular' cases (below):

a) \( s = \infty \) (cone surface) and \( e = 0 \) (convex hull of a ball and a point)
b) \( s = \infty \) (cone surface) and and \( e = 1 \) (cone base)
Example 1: no segments \((s = 0)\), one ellipse \((e = 1)\)

Joint numerical range \(W(A_1, A_2, A_3)\) of three Hermitian matrices,

\[
A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\]

forms a set with no segments and one ellipse in its boundary.
Example 2: one segment \((s = 1)\), one ellipse \((e = 1)\)

Joint numerical range \(W(A_1, A_2, A_3)\) of three Hermitian matrices,

\[
A_1 := \frac{1}{2} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

forms a set with one segment and one ellipse in its boundary.

Example 3: one segment \((s = 1)\), two ellipses \((e = 2)\)

Joint numerical range \(W(A_1, A_2, A_3)\) of three Hermitian matrices,

\[
A_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 := \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

forms a set with one segment and two ellipses in its boundary

Example 4: no segment \((s = 0)\), four ellipses \((e = 4)\)

Joint numerical range \(W(A_1, A_2, A_3)\) of three Hermitian matrices,

\[
A_1 := \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

forms a set with **no segment** and **four ellipses** in its boundary.
Ten classes of 3D numerical ranges labeled by

the number of segments $s$ and of ellipses $e$ in boundary
Concluding Remarks

- **Joint numerical range** of \( m \) Hermitian matrices of order \( N \) can be interpreted as
  a) **projection** of the \( N^2 - 1 \) dim. set of mixed states \( \mathcal{M}_N \) into \( \mathbb{R}^m \)
  b) set of possible **expectation values** of \( m \) quantum observables measured on several copies of the same quantum state \( \rho \) of size \( N \).

- **Joint numerical range** of \( m \) generic Hermitian matrices of order \( N \) forms an oval in \( \mathbb{R}^m \) and for large \( N \) is typically close to a ball (consequence of **Dvoretzki theorem**.)

- **Joint numerical range** of \( m = 3 \) Hermitian matrices of order \( N = 3 \) can divided into
  i) **five** classes of bodies with **no segments** and \( e = 0, 1, 2, 3, 4 \) **ellipses**
  ii) **three** classes of bodies with \( s = 1 \) **segment** and \( e = 0, 1, 2 \) **ellipses**
  iii) **two** classes of bodies with \( s = \infty \) **segments** (conical surface) and \( e = 0, 1 \) **ellipses**

  in the boundary
Concluding Remarks II

Research on **numerical range**, **numerical shadow** and their generalizations is

a) relevant for several problems of **quantum theory**

b) allows to establish links between various fields of **mathematics**:

- theory of convex sets (**Dvoretzki** theorem)
- theory of **approximations** (**B**–**splines**)
- theory of **probability** (**numerical shadow**)
- **random matrices**
- geometry of complex & real **projective spaces**
- ...

Visit our **Web page** on **numerical range** & **numerical shadow**

[www.numericalshadow.org](http://www.numericalshadow.org)
Some references (2011-2016)