GEOMETRY OF QUANTUM ENTANGLEMENT

some open problems

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Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos \frac{\vartheta}{2} |1\rangle + e^{i\phi} \sin \frac{\vartheta}{2} |0\rangle$$

Bloch sphere of $N = 2$ pure states

Space of pure states for an arbitrary $N$: a complex projective space $\mathbb{C}P^{N-1}$ of $2N - 2$ real dimensions.
Mixed quantum states

Set $\mathcal{M}_N$ of all mixed states of size $N$

$$\mathcal{M}_N := \{ \rho : \mathcal{H}_N \rightarrow \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \text{Tr}\rho = 1 \}$$

example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$$\rho = \sum_i a_i |\psi_i\rangle \langle \psi_i|$$
where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2 - 1}$.

How the set of all $N = 3$ mixed states looks like?

An 8 dimensional convex set with only 4 dimensional subset of pure (extremal) states, which belong to its 7 dim boundary
Euclidean Geometry of the set $\mathcal{M}^{(N)}$ of quantum states with respect to *Hilbert–Schmidt* distance:

$$D_{\text{HS}}(\rho, \sigma) := \sqrt{\text{Tr}(\rho - \sigma)^2}$$

Set $\mathcal{M}^{(N)}$ can be inscribed into an outsphere centred at the maximally mixed state $\rho_* = 1/N$ of radius $R_N = \sqrt{(N-1)/N}$. The insphere inscribed inside $\mathcal{M}^{(N)}$ has radius $r_N = 1/\sqrt{(N-1)N}$, so $R_2 = r_2$.

Hilbert–Schmidt distance leads to HS (flat) measure. For $N = 2$ (*one–qubit states*) we receive: the volume of the Bloch ball $V(2) = \pi\sqrt{2}/3$ and the area of the Bloch sphere of radius $R_2 = 1/\sqrt{2}$ reads $A(2) = 2\pi$. 

KŻ (IF UJ/CFT PAN)  Geometry of Quantum Entanglement  December 6, 2009  11 / 22
Volume and area of the set $\mathcal{M}^{(N)}$ of mixed states

with respect to the HS measure

Volume in $N^2 - 1$ dimensions

$$V(N) = \text{vol}(\mathcal{M}^{(N)}) = \sqrt{N(2\pi)^{N(N-1)/2}} \frac{\Gamma(1) \cdots \Gamma(N)}{\Gamma(N^2)} , \quad (1)$$

(Hyper) Area of this $N^2 - 2$ dimensional boundary

$$A(N) = \text{vol}(\partial \mathcal{M}^{(N)}) = \sqrt{N - 1}(2\pi)^{N(N-1)/2} \frac{\Gamma(1) \cdots \Gamma(N + 1)}{\Gamma(N)\Gamma(N^2 - 1)} . \quad (2)$$

Area/volume ratio $\gamma = Ar/V$

$$\gamma(\mathcal{M}^{(N)}) = r_N \frac{A}{V} = \frac{1}{\sqrt{(N - 1)N}} \sqrt{N(N - 1)(N^2 - 1)} = N^2 - 1 = d. \quad (3)$$

**Convex Body** = a convex, compact set in $\mathbb{R}^d$

A body $X$ has a **constant height** if

a) every boundary point of $X$ is contained in a face tangent to the inscribed ball of radius $r$ $\iff$

b) area/volume ratio is fixed, $\gamma(X) := rA/V = d = \text{dim}(X)$.

**Archimedean formula for the volume of a $d$-cone:**

Volume $V = Ah/d$, where $A$ stands for the area of its base and $h$ for its height $\implies$

A body of a constant height can be decomposed into cones of the same height!

**Set of mixed quantum states has a constant height**
Composed systems & entangled states

**bi-partite systems: \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \)**

- **separable pure states:** \( |\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \)
- **entangled pure states:** all states **not** of the above product form.

**Mixed states**

- **separable mixed states:** \( \rho_{\text{sep}} = \sum_j p_j \rho^A_j \otimes \rho^B_j \)
- **entangled mixed states:** all states **not** of the above product form.
Two-qubit mixed states

The maximal ball inscribed into $\mathcal{M}^{(4)}$ of radius $r_4 = 1/\sqrt{12}$ centred at $\rho_* = 1/4$ is separable!

$\rho^*$

thetrahedron of eigenvalues

K.Ż, P. Horodecki, M. Lewenstein, A. Sanpera, 1998
Two-qubit mixed states

Degree of entanglement: a distance to the closest separable state

K. Ž, M. Kuš, 2001
Positive partial transpose criterion: Two–qubit mixed states

The set of separable states of two–qubit system arises as an intersection of $\mathcal{M}^{(4)}$ and its mirror image with respect to partial transposition $T_A(\mathcal{M}^{(4)})$.

The set of two–qubit separable states has a constant height!

S. Szarek, I. Bengtsson, K.Ż, 2006
In 2005 P. Slater studied numerically the ratio $\Omega$ between the probabilities of finding a PPT state in the interior of $\mathcal{M}(N^2)$ and at its boundary,

$$\Omega \equiv \frac{p_V}{p_A} := \frac{V_{\text{PPT}}}{V_{\text{tot}}} = \frac{V_{\text{PPT}} A_{\text{tot}}}{V_{\text{tot}} A_{\text{PPT}}}.$$  \hspace{1cm} (4)

Area of the PPT part of the boundary of the set of mixed states $A_P = A_{\text{PPT}}/2$ due to reflection symmetry. Since the set of PPT states has the same constant height $r$ as the set of all states then

$$rA_{\text{tot}}/V_{\text{tot}} = rA_{\text{PPT}}/V_{\text{PPT}}.$$  

Hence we find

$$\Omega = \frac{V_{\text{PPT}} A_{\text{tot}}}{V_{\text{tot}} A_{\text{PPT}}/2} = 2.$$  \hspace{1cm} (5)

This statement concerns the PPT property for any $N \times N$ system, and for two–qubit system it concerns also the relative probability of finding a separable state.

S. Szarek, I. Bengtsson, K.Ż, 2006
Open problems - mixed states: volumes

Find if there is any symmetry governing the PPT reflection of the set of mixed states, which would allow us to established another relation between between $V_{\text{PPT}}$ and $V_{\text{tot}}$.

Compute the exact value of the ratio $V_{\text{PPT}}/V_{\text{tot}}$ which would give the exact value of the volume Hilbert–Schmidt volume of the set of two-qubit separable states.

Find a similar relation for two quNit systems which yields the relative volume of the PPT states - check how it depends on $N$. 
Open problem - visualisation

What is a 'best possible' 3-dimensional model of the 8-dimensional set of mixed states of \( N = 3 \)?

Is it the convex hull of a tennis ball?
Two qubits: foliation of complex projective space $\mathbb{C}P^3$ into leaves of equal FS distance to the set of product states $\mathbb{C}P^1 \times \mathbb{C}P^1$.

A generic 5–D leaf of an intermediate entanglement $\omega = C/2 > 0$ has the local structure of $U(2)/[U(1)]^2 \times \mathbb{R}P^3$. 
Open problems - pure states: topology

Pure states of bipartite system

Two qubits: The orbit of states locally equivalent to the maximally entangled Bell state has 3 dimensions and the topology of real projective space $\mathbb{R}P^3$.

Two quNits: The orbit of states locally equivalent to maximally entangled state has $N^2 - 1$ dimensions and the topology of a coset space $U(N)/U(1)$.

Pure states of multi-partite system

- three qubits - pure states: What is the topology of the set of pure states locally equivalent to  
  a) GHZ state, $|\psi_{\text{GHZ}}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$,  
  and  
  b) W–state, $|\psi_{\text{W}}\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ ?

Similar questions for – four qubits - pure states ...