Entropic Uncertainty Relations

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in collaboration with

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Heisenberg uncertainty relation (1927)

Formulation of Kennard (1927) for the product of variances of position and momentum ($\hbar = 1$)

$$\Delta^2 x \Delta^2 p \geq \frac{1}{4}.$$ 

A more general formulation of Robertson (1929)

for arbitrary operators $A$ an $B$. Let $\Delta^2 A = \langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2$ be the variance of an operator $A$. Then for any state $|\psi\rangle$

$$\Delta^2 A \Delta^2 B \geq \frac{1}{4} |\langle \psi | AB - BA | \psi \rangle|^2.$$ 

As $[x, p] = xp - px = i$ the latter form implies the former bound.
Continuous case

Define continuous (Boltzmann–Gibbs) entropies:

\[ S(x) = -\int dx |\psi(x)|^2 \ln |\psi(x)|^2 \]

and

\[ S(p) = -\int dp |\psi(p)|^2 \ln |\psi(p)|^2. \]

Then

\[ S(x) + S(p) \geq \ln(e\pi). \]

Białynicki-Birula, Mycielski (1975) and Beckner, (1975)

generalizations for Rényi entropies,

\[ S_\alpha(x) := \frac{1}{1-\alpha} \ln \left( \int dx |\psi(x)|^{2\alpha} \right) \]

Our work is dedicated to Prof. Iwo Białynicki-Birula, (born June 14, 1933), on the occasion of his 80–th birthday
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State $|\psi\rangle = \sum_{i}^{N} a_{i} |i\rangle = \sum_{j} b_{j} |\beta_{j}\rangle$ is expanded in the eigenbases of operators $A$ and $B$, related by a unitary matrix $U_{ij} = \langle i | \beta_{j} \rangle$.

Let Shannon entropies in both expansion be

$S^{A}(\psi) = -\sum_{i=1}^{N} p_{i} \ln p_{i} = S(p)$ with $p_{i} = |a_{i}|^{2}$, $\sum_{i} p_{i} = 1$ and

$S^{B}(\psi) = -\sum_{j=1}^{N} q_{j} \ln q_{j} = S(q)$ with $q_{j} = |b_{j}|^{2}$, $\sum_{j} q_{j} = 1$.

Let $c(A, B) = \max_{ij} |U_{ij}|$. Then for any state $|\psi\rangle \in \mathcal{H}_{N}$ we have

$S^{A}(\psi) + S^{B}(\psi) \geq -2 \ln[(1 + c)/2]$

Deutsch, (1983), later improved

$S^{A}(\psi) + S^{B}(\psi) \geq -2 \ln c$

by Maassen, Uffink, (1988),
Example: the Fourier matrix $F_N$

Unitary matrix which defines the second basis

$$U_{jk} = (F_N)_{jk} := \frac{1}{\sqrt{N}} \exp(i \frac{2\pi jk}{N}) \quad \text{with} \quad j, k = 0, 1, \ldots, n - 1.$$ 

then $c = \max_{jk} |U_{jk}| = 1/\sqrt{N}$.

The bound of Maassen–Uffink gives

$$S(p) + S(q) \geq -2 \ln c = \ln N$$

If $|\psi\rangle = (1, 0, \ldots, 0)$ then $S_A = 0$ and $S_B = \ln N$ so bound is saturated...

The same bound holds for any unitary complex Hadamard matrix $H$, for which $|H_{ij}|^2 = 1/N$ for all $i, j = 1, \ldots N$.

see online Catalogue of Complex Hadamard matrices

http://chaos.if.uj.edu.pl/~karol/hadamard
Wawel Castle, **Cracow**, Poland
Warm up: a classical analogue of M–U bounds

Classical map: stochastic matrix $T$

Let $p$ be a probability vector of size $N$, and its image is $q = Tp$ where $T$ is a **stochastic** matrix, $T_{ij} \geq 0$ and $\sum_i T_{ij} = 1$.

We show that the sum of both entropies is bounded from below by a function of the **largest** entry of the transition matrix $T$,

$$S(p) + S(Tp) \geq -\ln \kappa,$$

where $\kappa = \max_{ij} T_{ji}$

Proof is based on an inequality by Słomczyński (2002) concerning the mean entropy of a stochastic matrix $T$ and a probability vector $p$,

$$S^{(p)}(T) \leq S(Tp) \leq S^{(p)}(T) + H(p).$$

Here $S^{(p)}(T) = \sum_i p_i S(t_i)$, while $S(T) = \sum_i \frac{1}{N} S(t_i)$,
Optical effects at the Market Square in Cracow, Poland
Back to quantum case...

The **Maassen–Uffink** bounds are strong, but not optimal.

**How to improve them??**

A useful tool: the **Rényi entropy**

\[ S_\alpha(p) = \frac{1}{1 - \alpha} \ln \left( \sum_i p_i^\alpha \right), \]

where \( \alpha \geq 0 \) is a free parameter.

In the limit \( \alpha \to 1 \) the **Rényi entropy** tends to the **Shannon entropy**,

\[ \lim_{\alpha \to 1} S_\alpha(p) = S(p) =: S_1(p). \]

The **Rényi entropy** \( S_\alpha(p) \) is a monotonously decreasing function of the Renyi parameter \( \alpha \).

A simple relation \( S_\infty(p) = -\ln(p_{\text{max}}) \) allows us to rewrite the **Maassen–Uffink** bound:

\[ S_1(p) + S_1(q) \geq \max_j \left[ S_\infty(v_j) \right], \]

where columns of the **transition matrix** read \( v_j = T_{jm} = |U_{jm}|^2, \)

\( j = 1, \ldots, N \).
A first observation

a) Take a base vector $|\psi\rangle = |k\rangle$, so that $p = (0, \ldots, 1, 0, \ldots 0)$ and $S(p) = 0$.
b) Then $U|\psi\rangle = \sum_j U_{ij}|j\rangle$, so $q_j = |\langle \beta_j|\psi\rangle|^2 = |U_{jk}|^2$
and $q = (|U_{1k}|^2, \ldots, |U_{N,k}|^2) = v_k$
c) In this case one has: $S(p) + S(q) = 0 + S(q) = S(v_k)$

A first attempt (to use several elements of unitary $U$)

The above observation suggests one to analyze a relation (with $\alpha = 1$)

$$S_1(p) + S_1(q) \geq \max_i \left[ S_1(v_i), S_1(v'_i) \right]$$

We use here the columns $v_i = T_{ij}$, and the rows, $v'_i = T_{ji}$,
of the unistochastic transition matrix $T_{ij} = |U_{ij}|^2$

This relation is false, but its weaker form with $\alpha = 2$

$$S_1(p) + S_1(q) \geq \max_i \left[ S_2(v_i), S_2(v'_i) \right]$$

holds for $N = 2$ and $N = 3$ (numerical results)

However, it fails for $N \geq 4$!
Wawel Castle, Cracow, Poland
Another approach: Key ingredients used

A) An algebraic tool:  **Majorization**

Consider two probability vectors of length $N$ ordered decreasingly, $x = (x_1 \geq x_2 \geq \ldots x_N \geq 0)$ and $y = (y_1 \geq y_2 \geq \ldots y_N \geq 0)$.

The vector $x$ is called to be **majorized** by $y$, written $x \prec y$, if
\[ \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i, \quad \text{for} \quad m = 1, \ldots N - 1 \]

**Majorization** $x \prec y$ implies inequalities for **Renyi entropies**
\[ S_\alpha(x) \geq S_\alpha(y) \]
(and other **Schur–concave** functions)

B) **Bi-entropy and product probability vectors**

Let $p \otimes q = (p_1 q_1, p_1 q_2, \ldots, p_1 q_N, \ldots p_N q_N)$

\[ \text{denotes a product probability vector of size } N^2. \]

Then the bientropy reads
\[ S_\alpha(p) + S_\alpha(q) = S_\alpha(p \otimes q). \]

To arrive at an **entropic uncertainty relation**

we need to find a vector $R$ majorizing $p \otimes q$. 
First step, \((k = 1)\): The bound of Deutsch reused

**Deriving his uncertainty relation** Deutsch proved in 1983 that

\[
\max_{\psi, i, j} p_i q_j = \max_{\psi, i, j} \left( \frac{p_i + q_j}{2} \right)^2 = \left( \frac{1 + c}{2} \right)^2 =: R_1,
\]

where as before \(c = \max_{ij} |U_{ij}|.\)

**A direct implication**

The Rényi entropies are **Schur-concave**, so

\[
(p \otimes q) \prec (R_1, 1 - R_1, 0, \ldots 0) =: (Q_1, Q_2, \ldots, Q_{N^2}).
\]

which implies a first bound for the entire family of Rényi entropies \(S_\alpha\)

\[
S_\alpha(p) + S_\alpha(q) \geq S_\alpha(Q) = \frac{1}{1 - \alpha} \ln \left[ R_1^\alpha + (1 - R_1)^\alpha \right].
\]

For \(\alpha = 1\) we get a **stronger** result than the original bound of Deutsch.
Example: Bounds for $N = 2$ orthogonal matrix $O$

For an orthogonal matrix of size $N = 2$

$$O(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

we plot bounds for the **Rényi binetropy**

$$S_\alpha(p) + S_\alpha(q) \geq B = S_\alpha(R_1, 1 - R_1)$$

Bounds for **equivalent unitary matrices** are equal

Two unitary matrices $U$ and $V$ are equivalent, $U \sim V$, if there exist permutation matrices $P_1, P_2$ and diagonal unitary matrices $D_1, D_2$, such that

$$V = P_1 D_1 U D_2 P_2$$

**Lemma 1.** Massen-Uffink and Majorization bounds for **equivalent** unitaries are the same.

**Lemma 2.** Any $N = 2$ unitary matrix is **equivalent** to an **orthogonal** matrix,

$$U(2) \ni U \sim O(\theta).$$
The best hotel in **Cracow** ...
Entering **Cracow**, Banana Shire, Queensland
ALL HEAVY VEHICLES TO PROCEED TO THE BOOM GATE AND REPORT TO OFFICE

CRACOW GOLD MINE RECEPTION
Second step \((k = 2)\)

**k = 2: norms of 2-subvectors of unitary \(U\)**

To improve the former result we look for stronger majorization of the type

\[
(p \otimes q) \prec (R_1, R_2 - R_1, 1 - R_2, 0, \ldots 0).
\]

(1)

Consider the **longest 2–sub–vector** of unitary \(U\) and denote its norm by

\[
s_2 = \max \left\{ \max_{i,j_1,j_2} \sqrt{|U_{i_1j_1}|^2 + |U_{i_2j_2}|^2}, \max_{i_1,i_2,j} \sqrt{|U_{i_1j}|^2 + |U_{i_2j}|^2} \right\}
\]

Our main theorem implies that the above **majorization relation** with \(R_2 = \left(\frac{1+s_2}{2}\right)^2\) holds!

Example: On orthogonal matrix \(U \in U(4)\) with entries truncated to two decimal digits

\[
\begin{bmatrix}
0.19 & 0.50 & -0.64 & 0.55 \\
-0.62 & 0.54 & -0.21 & -0.52 \\
0.52 & -0.21 & -0.54 & -0.62 \\
-0.55 & -0.64 & -0.50 & 0.19 \\
\end{bmatrix}
\]
Example: Bounds for a family of $N = 3$ unitary matrices

Consider an $N = 3$ permutation matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

EUR for unitary matrices $P^\beta$, where $\beta \in [0, 1]$ for **Shannon binetropy**:

$S(p) + S(q) \geq S(R_1, R_2 - R_1, 1 - R_2) \geq S(R_1, 1 - R_1)$.

**Majorization EUR: a comparison with Maassen - Uffink**

Bistochastic triangle $B = \text{Convhull}(P, P^2, P^3 = 1)$. Corresponding unitary matrices, $|U_{ij}|^2 = B_{ij}$, belong to 3-hypocycloid. **Blue region:** the **MU bound** is weaker than **majorization** bound.
Third step, ($k = 3$)

$k = 3$: spectral norms of truncations of unitary $U$

For any matrix $X$ we define its spectral norm by the largest singular value,

$$\|X\| = \sigma_{\text{max}}(X) = \max[\sqrt{\lambda(XX^\dagger)}].$$

Let $A_{m,n}$ denote the maximal $m \times n$ truncation of $U$ (with largest norm).

Define $s_3 := \max\{|\|A_{1,3}\||, |\|A_{2,2}\||, |\|A_{3,1}\||\}$ and $R_3 := (\frac{1+s_3}{2})^2$.

Then a stronger majorization relation holds

$$(p \otimes q) \prec (R_1, R_2 - R_1, R_3 - R_2, 1 - R_3, 0, \ldots 0). \quad (2)$$

Example: the same orthogonal matrix $U \in U(4)$ with entries truncated to two decimal digits

$$
\begin{bmatrix}
0.19 & 0.50 & -0.64 & \text{0.55} \\
-0.62 & 0.54 & -0.21 & \text{-0.52} \\
0.52 & -0.21 & -0.54 & \text{-0.62} \\
-0.55 & -0.64 & -0.50 & 0.19
\end{bmatrix}
$$
Example: Bounds for a family of $N = 4$ unitary matrices

Consider an $N = 4$ permutation matrix $P_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

**Entropic Uncertainty Relations** for unitary matrices $P_4^\beta$, where $\beta \in [0, 1]$ for Shannon bitentropy:

$S(p) + S(q) \geq S(R_1, R_2 - R_1, R_3 - R_2, 1 - R_3) \geq S(R_1, R_2 - R_1, 1 - R_2) \geq S(R_1, 1 - R_1)$.

with $R_k = \left(\frac{1 + s_k}{2}\right)^2$
The main theorem: Majorization EUR

Let \( k = 1, \ldots N - 1 \): spectral norms of all submatrices of unitary \( U \)

Let \( A_{m,n} \) denote the maximal \( m \times n \) submatrix of \( U \).

Define \( s_k := \max\{||A_{1,k}||, ||A_{2,k-1}||, \ldots, ||A_{k-1,2}||, ||A_{k,1}||\} \).

We have \( s_k \geq s_{k-1} \) and \( R_k := \left(\frac{1+s_k}{2}\right)^2 \geq R_{k-1} \).

Theorem: For any unitary \( U \) of order \( N \)

the following majorization relation holds:

\[(p \otimes q) \prec (R_1, R_2 - R_1, \ldots, R_{N-1} - R_{N-2}, 1 - R_{N-1}) =: Q. \quad (3)\]

This implies an explicit majorization entropic uncertainty relation

\[S_\alpha(p) + S_\alpha(q) \geq S_\alpha(Q) = \frac{1}{1 - \alpha} \ln \sum_{i=1}^{N^2} Q_i^\alpha.\]
The main theorem: a sketch of the proof

Bound of sum $R_k$ of $k$ largest components of vector $Q$ majorizing $p \otimes q$, which contains terms $(p_1 q_1, p_1 q_2, p_2 q_1, \ldots, p_N q_N)$ ordered decreasingly

$$\max _{1 \leq m \leq k-1} \left( \sum _{i=1} ^{k-m} p_i \right) \left( \sum _{j=1} ^m q_j \right) \leq \max _{1 \leq m \leq k-1} \frac{1}{4} \left( \sum _{i=1} ^{k-m} p_i + \sum _{j=1} ^m q_j \right) ^2.$$  \hspace{1cm} (4)

and use the fact $\sqrt{xy} \leq (x + y)/2$.

To complete the proof we use an algebraic lemma

Let $|1\rangle, |2\rangle, \ldots, |m\rangle$ and $|a_1\rangle, |a_2\rangle \ldots |a_n\rangle$ be two orthonormal sets of vectors, then

$$\max _{|\psi\rangle} \left( \sum _{i=1} ^m |\langle i|\psi\rangle|^2 + \sum _{j=1} ^n |\langle a_j|\psi\rangle|^2 \right) = 1 + \sigma_1(A),$$

where $\sigma_1(A) = ||A||$ is the leading singular value of rectangular sub–unitary matrix $A = \{A_{ij}\}_{i=1,j=1} ^{m,n}$ for $A_{ij} = \langle a_i|j\rangle$.

One can find $|\psi\rangle$ such that both terms above are equal $\iff x = y$ and bound (4) can be saturated.
Market Square, Cracow, Poland
Entropic Uncertainty Relations for a generic unitary $U$ of size $N$

i) Asymptotic Massen–Uffink bound

$$S^A(\psi) + S^B(\psi) \geq -\ln |U_{\text{max}}|^2$$

a) the largest squared element of a normalized complex random vector $v$ of length $N$ behaves as

$$|v_{\text{max}}|^2 \approx_{a.s.} \frac{\log N}{N}$$

b) the largest squared element of a random unitary matrix $U$ of size $N$ behaves as

$$|U_{\text{max}}|^2 \approx_{a.s.} 2\frac{\log N}{N} \quad \text{(Jiang 2005)}$$

This implies an asymptotic behaviour

$$S^A(\psi) + S^B(\psi) \geq B_{MU} \approx_{a.s.} \log N - \log 2 - \log \log N$$
ii) **Asymptotic Majorization bounds**

We need estimations for the size of $|A_{m,n}|^2$ – the **maximal** $m \times n$ truncation of $U$

**Proposition:** For a random unitary $U$ of size $N$ the squared norm of its largest $m \times n$ truncation ($n, m$ fixed), behaves as

$$|A_{m,n}|^2 \approx a.s. \ (m + n) \frac{\log N}{N}$$

(the same result of truncations of the same semiperimeter $m + n$)
ii) Asymptotic Majorization bounds

We need estimations for the size of $|A_{m,n}|^2$ – the maximal $m \times n$ truncation of $U$

**Proposition:** For a random unitary $U$ of size $N$ the squared norm of its largest $m \times n$ truncation ($n, m$ fixed), behaves as

$$|A_{m,n}|^2 \approx_{a.s.} (m + n) \frac{\log N}{N}$$

(the same result of truncations of the same semiperimeter $m + n$)

This implies an asymptotic estimation

$$S^A(\psi) + S^B(\psi) \geq B_{major} \approx_{a.s.} \frac{3}{4} \log N + \frac{2 \log 2 - 1}{4}$$

which is stronger than $B_{MU}$ for a small dimension $N$, but becomes weaker for $N \approx 10000$. 
Concluding remarks

- A classical analogue of Maassen-Uffink bound is derived.
- Main result: **Majorization Entropic Uncertainty Relation** derived for any unitary $U \in U(N)$ in terms of **spectral norms** of its **maximal submatrices**.
- **Majorization** EUR for equivalent unitary matrices are **equal**.
- For a random unitary matrix $U$ the **majorization EUR** are stronger than Maassen–Uffink relations with probability $0.815$ for $N = 2$ and $0.971, 0.972, 0.983, 0.990$, for $N = 3, 4, 5, 6$ respectively.
- However, making use of **concentration of measure** and estimations for the norms of **maximal truncations** of $U$ we show that asymptotically (large $N$) **MU bounds** become stronger...
- Analogous results obtained independently by Friedland, Gheorghiu and Gour, preprint arXiv:1304.6351
Cracow (Poland !) with the Wawel Castle and the Tatra mountains in the background.