Quantum chaos: An entropy approach

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A new definition of the entropy of a given dynamical system and of an instrument describing the measurement process is proposed within the operational approach to quantum mechanics. It generalizes other definitions of entropy, in both the classical and quantum cases. The Kolmogorov–Sinai (KS) entropy is obtained for a classical system and the sharp measurement instrument. For a quantum system and a coherent states instrument, a new quantity, coherent states entropy, is defined. It may be used to measure chaos in quantum mechanics. The following correspondence principle is proved: the upper limit of the coherent states entropy of a quantum map as $\hbar \to 0$ is less than or equal to the KS-entropy of the corresponding classical map.

"Chaos umpire sits,
And by decision more imbroils the fray
By which he reigns: next him high arbiter
Chance governs all."
John Milton, Paradise Lost, Book II

I. INTRODUCTION

Quantum chaos$^{1,2}$ is a controversial subject. Some physicists regard it as a theory, established to such an extent that it may be presented to the general audience.$^3$ Others treat it merely as “a new style in Emperor’s clothing” and seem to doubt the possibility of fitting it into the present-day picture of quantum mechanics.$^4-10$ The former stress the differences between quantum systems with classical chaotic and regular counterparts such as, for instance, the type of the distribution of nearest-neighbor energy level spacings, which can be Poisson or skew normal (Wigner), respectively. The latter believe that no quantum time evolution can be recognized as chaotic if one looks at the problem from the algorithmic complexity theory point of view, which they regard as the most relevant to the case.

Indisputably, two problems seem to be open and, in a sense, embarrassing (see also Refs. 11 and 12):

1. The lack of a universally accepted definition of quantum chaos and the lack of quantities which may be used to measure chaos, such as entropy or Lyapunov exponents in the classical case.

2. An inconsistency with the Correspondence Principle: classical chaotic systems and their quantum mechanical counterparts seem to show qualitatively different behavior, even in the limit as $\hbar \to 0$.

In the present paper we put forward a rigorous mathematical framework to discuss and, hopefully, solve these problems.

Let us start from some general considerations. Ford$^4$ posed the following basic question: What is chaos and what does it mean? We believe that the most fundamental feature of a chaotic system is unpredictability. The precise mathematical meaning of this word may be given within the algorithmic complexity theory or the information theory. However, even if we consent to this
definition, the following question still remains: What are the quantities whose evolution we try in vain to predict? Two possible answers should be considered: unpredictability may concern either the states of the system or the outcomes of a measurement. Even though evidently connected with one another, these two approaches do not coincide. The difference is less important in the classical case, where we can, at least theoretically, measure the state of a system (e.g., the positions and momenta of classical objects) with an arbitrary accuracy and assume that the process of measurement does not alter the dynamics of the system. However, even in that case, the methods of extracting the entropy or the Lyapunov exponents from the equation of motion and from the analysis of the data (the time series) differ considerably. They are, nevertheless, believed to lead to the principally identical results. In the quantum case the situation is entirely different. The time evolution of the states of a given quantum system described in the simplest case by wave functions is almost periodic and thus predictable, at least for finite particle number, spatially bounded and undriven quantum system.\cite{4,5,7,8} In contrast, successive outcomes of a measurement can form chaotic and unpredictable sequences, and, as we see further, they really do!

One can object to the former approach (as Ford did), distinguishing between intrinsic quantum chaotic behavior and chaotic behavior occurring in a quantum system due to interactions with classical measuring devices. However, “intrinsic quantum chaos” is a rather vague notion, and, if we subscribe to Ford’s point of view as to a general rule, we might be forced to classify one and the same system once as regular and another time as chaotic, depending on the particular mathematical framework we used. For instance, we can describe the dynamics of a classical mechanical system as the evolution in the phase-space governed by the canonical equations of motion or as the evolution in the space of smooth densities governed by the Liouville equation. Now, if we consider any mixing system, e.g., the geodesic motion of a mass point on a compact surface of negative curvature or the Arnold cat map, it becomes evident that the chaotic and unpredictable behavior of point trajectories coexists with a regular and predictable behavior of densities, which tend weakly to the uniform density (see, however, Ref. 6).

Quantum mechanics yields another example. As we have already remarked, the time evolution of a quantum state may be described by Schrödinger equation, which is linear and so, in a sense, leads to predictable dynamics. However, the same system can be equivalently described by the equations of Bohmian mechanics, which are nonlinear, and hence admit any kind of chaotic behavior (see Ref. 13 and Ref. 14, §6.10.2).

Summing up—we believe that the approach linking chaos with the unpredictability of the measurement outcomes is the right one in the quantum case (see also Ref. 12). In order to measure this unpredictability we introduce the notion of entropy, which generalizes other definitions, in both the classical and quantum case. Several attempts to introduce the notion of entropy into quantum mechanics have been made,\cite{1,15,39} however, none of them might be applied to define quantum chaos rigorously. In this work we try to bridge the gap by introducing a new definition of quantum entropy based on coherent states. This entropy is a function of the entire dynamical system and the measurement process. Hence, it should be distinguished from von Neuman entropy\cite{17,29} given by $-\text{Tr} \rho \ln \rho$, which describes the properties of a single state $\rho$, and from Ingarden–Urbanik entropy (A-entropy), which describes the properties of a quantum state after a single measurement.\cite{15}

The chief points of the general framework for the study of quantum chaos we would like to propose are the following.

(1) The notions of chaos and regularity may be attributed to the pair consisting of a quantum system and an instrument measuring an observable, rather than to the quantum system alone.

(2) We assume that a large number of successive measurements are performed on the evolving system by means of the measuring instrument and that the motion of the system is perturbed by the measurement process. Then the complete dynamics can be described by a quantum stochastic process (QSP). In special cases the process is Markovian and the evolution of the measurement outcomes can be described by a Markov operator.
(3) For the quantum stochastic process we define a non-negative quantity, *quantum entropy*, which evaluates the degree of randomness of the sequences of the measurement outcomes. We propose the division of quantum entropy into two components: *quantum measurement entropy* and *quantum dynamical entropy*. Each of them measures a different kind of randomness: the first, that coming from the process of measurement, and the second, that connected with the underlying dynamics of the system.

(4) To compare the behavior of quantum mechanical systems and their classical counterparts we study the combined dynamics arising as a result of the interaction between the evolving quantum system and the *generalized coherent states instrument* which is related to an approximate or fuzzy measurement in quantum mechanics. The distribution of the outcomes of the measurement performed by means of this instrument is the *generalized Husimi representation* of the state of the system at the instant after the measurement. The QSP associated with the combined evolution of the system repeatedly measured by this apparatus is Markovian and the corresponding Markov operator has a smooth kernel.

We call quantum entropy connected with this QSP the *coherent states quantum entropy* or, briefly, *CS-quantum entropy*. Its dynamical component may be used to measure the degree of the stochasticity connected with the unitary dynamics and positivity of this component may be accepted as a rigorous definition of *quantum chaos*. Loosely speaking we can say that the quantum system is *chaotic* if the unitary dynamics produces some *additional unpredictability* beyond that induced by the measurement process. We try to quantify this idea by introducing dynamical CS-entropy.

(5) We apply the results of the theory of random perturbations of dynamical systems to state the *correspondence principle for entropy*. Omitting several technical assumptions, it can be expressed as follows:

The upper limit of the CS-quantum entropy of a quantized classical dynamical system as $\hbar \to 0$ is less than or equal to the Kolmogorov–Sinai (KS) entropy of this system.

We conjecture that if the classical dynamical system is hyperbolic and transitive (e.g., the cat map), then the CS-quantum entropy in question gives the KS entropy of this system in the semiclassical limit ($\hbar \to 0$).

(6) In several cases [e.g., for SU(N) coherent states] we can show that measurement CS-quantum entropy tends to zero as $\hbar \to 0$. Then the correspondence principle also holds for the dynamical CS-quantum entropy. In other words, in this case, the two limits, the time limit ($t \to \infty$) and the semiclassical limit ($\hbar \to 0$), may commute. This feature differs considerably our model from other approaches to the notion of quantum entropy.

The present paper is organized in the following way.

In Sec. II we recall the basic facts of the operational approach to quantum mechanics. Since in our description of quantum chaos we take into consideration the measurement process and want to compare classical and quantum systems, the operational approach seems to be the most natural and convenient one. In Sec. III we introduce quantum entropy, quantum measurement entropy, and quantum dynamical entropy, study their properties and compare our definition with other concepts of entropy existing in the literature. In Sec. IV we analyze in detail quantum entropies connected with generalized coherent states instruments which seem to be the most important from “quantum chaotic” point of view. In Sec. V we show how to formulate the corresponding principle for entropy, applying the results of the theory of random perturbations of dynamical systems. In Sec. VI we illustrate our approach by an example of a quantized classical map on the sphere $S^2$. Finally, Sec. VII contains concluding remarks and a list of open problems.
II. OPERATIONAL APPROACH TO QUANTUM MECHANICS

The operational approach to quantum theory was developed in an axiomatic form by many authors in the late 60s and early 70s. In this section we mainly follow Davies and Lewis’ formulation,44-46 partially suited to our needs. Their approach seems to be broad enough to cover all the applications we refer to. Nevertheless, one could also use the more general approach of Gudder47 based on the notion of a convex structure or the approach based on the general theory of measurement in quantum mechanics.48,49

A. State space and phase space

We define a state space as a pair $(V, K)$, where

(i) $V$ is a real Banach space with the norm $\| \|$;
(ii) $K$ is a closed cone in $V$;
(iii) if $u, v \in K$, then $\|u\| + \|v\| = \|u + v\|$;
(iv) if $u \in V$ and $\epsilon > 0$, then there exist $u_1, u_2 \in K$ such that $u = u_1 - u_2$ and $\|u_1\| + \|u_2\| < \|u\| + \epsilon$.

If $(V, K)$ is a state space, then there exists the unique positive linear functional $\tau : V \to \mathbb{R}$ such that $\tau(u) = \|u\|$ for $u \in K$ and $\tau(u) \leq \|u\|$ for $u \in V$. We say that $u \in K$ is a state if $\tau(u) = 1$.

The following examples of state spaces seem to be the most important ones.

(A) Classical mechanics—Let $X$ be a locally compact Hausdorff space. Let $V$ be the space of all countable additive regular set functions on the Borel $\sigma$-algebra of $X$ endowed with the total variation norm. Define $K$ to be the set of all non-negative measures from $V$. Then the states will be just the probability measures on $X$.

(B) Hilbert space quantum mechanics—Let $\mathcal{H}$ be a Hilbert space. Let $V = \mathcal{H}^*$ be the space of all self-adjoint trace class operators on $\mathcal{H}$ endowed with the trace norm. Define $K$ to be the set of all positive operators from $V$. In this case $\tau(A) = \text{tr}(A)$ for all $A \in V$.

(C) C*-algebra quantum mechanics—Let $A$ be a C*-algebra. Let $V$ be the space of all bounded self-adjoint linear functionals on $A$ with the dual space norm. Set $K = \{\omega \in V : \omega(a^*a) \geq 0 \text{ for all } a \in A\}$. In fact, this example contains the two preceding ones:45 (A) if $A$ is taken to be the algebra of all continuous functions on $X$ vanishing at infinity, and (B) if we take $A$ to be the algebra of all compact operators on $\mathcal{H}$.

By a phase space (a value space or a space of experimental outcomes) we mean an arbitrary measurable space $(\Omega, \Sigma)$, where $\Omega$ is a set representing possible outcomes of a measurement and $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$.

B. Observables and instruments

The notions of observable and instrument play the central role in the operational formulation of quantum mechanics. An instrument describes both the probability distribution of the outcomes of a measurement and the change of a state due to the measurement.

Let $(\Omega, \Sigma)$ be a phase space, let $(V, K)$ be a state space, and let $V^*$ be the Banach space dual to $V$. We introduce a partial order in $V^*$ by defining $\phi \succeq \psi$ if and only if $\phi(u) \geq \psi(u)$ for all $u \in K$. An effect is a map $\phi \in V^*$ such that $0 \leq \phi \leq \tau$. We denote the set of all effects by $\mathcal{E}$. This set is convex and compact in the weak *-topology of $V^*$.

We say that $x : \Sigma \to \mathcal{E}$ is an observable if $x$ is an effect-valued measure such that $x(\Omega) = \tau$. If $E \in \Sigma, u \in K$, and $\tau(u) = 1$, then $x(E)u$ can be interpreted as the probability that the result of the measurement of the physical quantity represented by $x$ prepared in a state $u$ belongs to the set $E$.

In the case of Hilbert space quantum mechanics, effects can be identified with bounded operators $A$ such that $0 \leq A \leq 1$ by the formula $\phi_A(W) = \text{tr}(AW)$. Analogously, observables can be treated as positive-operator-valued (POV) measures. This generalizes the classical notion of an observable as a projection-valued (PV) measure.
We define an operation as a positive linear operator $T: V \rightarrow V$ satisfying $0 \leq \tau(Tu) \leq \tau(u)$ for every $u \in K$. The set of all operations will be denoted by $\mathcal{O}$. By an operation-valued measure (OVM) on a phase space we mean a map $\mathcal{F}: \Sigma \rightarrow \mathcal{O}$ such that, if $\{E_n\}$ is a sequence of disjoint sets in $\Sigma$, then $\mathcal{F}(\bigcup E_n) = \sum \mathcal{F}(E_n)$, where the sum converges in the strong operator topology.

Let $\mathcal{F}: \Sigma \rightarrow \mathcal{O}$ be an OVM. We say that $\mathcal{F}$ is an instrument if $\tau(\mathcal{F}(\Omega)u) = \tau(u)$ for every $u \in V$.

The following is the interpretation of this notion. Let $\mathcal{F}$ be an instrument, $E \in \Sigma$ and $u \in K$. If $u$ is the state of the system at the instant before the measurement and the instrument $\mathcal{F}$ determines a value in the set $E$, then the output state is given by $\mathcal{F}(E)u/\tau(\mathcal{F}(E)u)$. For every instrument $\mathcal{F}$ there exists the unique observable $x_\mathcal{F}$ such that $\tau(\mathcal{F}(E)u) = x_\mathcal{F}(E)u$ for all $E \in \Sigma$ and $u \in K$. Note, however, that two different instruments may correspond with the same observable.

We give now several examples of instruments that correspond to different kinds of measurement in classical and quantum mechanics. We start from the Lüders-von Neumann measurement [example (D)]. To describe “approximate,” “fuzzy,” or “unsharp” measurement in quantum mechanics we shall study generalized coherent states [examples (E) and (F)]. Coherent states, being a “natural language of quantum theory,” play special role in our approach. We shall use them in Sec. IV to give a new definition of quantum entropy. For an exhaustive review of the subject, see Refs. 50–54. Next, we analyze sharp and approximate measurements in classical mechanics [example (G)]. The notion of a nuclear instrument [examples (H) and (J)] generalizes the previous ones. Finally, we define a transformed instrument [example (K)].

(D) Hilbert space quantum mechanics: the Lüders–von Neumann instrument

Let $\mathcal{H}$ be a Hilbert space, let $\Omega = \{1, \ldots, N\}$ or $\Omega = \mathbb{N}$, and let $\{P_i\}_{i \in \Omega}$ be a family of orthogonal projections on $\mathcal{H}$ such that $I = \sum_{i \in \Omega} P_i$. Then $\mathcal{F}$ defined by the famous “collapse of wave function formula,”

$$\mathcal{F}(E)\rho = \sum_{i \in E} P_i \rho P_i,$$

for a set $E \subseteq \Omega$ and a trace class operator $\rho$, is an instrument. The corresponding observable is given by

$$x_\mathcal{F}(E)\rho = \sum_{i \in E} \text{tr}(P_i \rho)$$

for $E \subseteq \Omega$ and $\rho \in \mathcal{F}(\mathcal{H})$.

(E) Hilbert space quantum mechanics: generalized coherent states

Let $\mathcal{H}$ be a Hilbert space, let $\Omega$ be a topological space, let $\Sigma$ be the Borel $\sigma$-algebra of $\Omega$, and let $m$ be a measure on $(\Omega, \Sigma)$. Let $\{P_a\}_{a \in \Omega}$ be a family of finite-dimensional projections on $\mathcal{H}$ such that (i) the map $a \rightarrow P_a$ is strongly continuous and (ii) $\int_{\Omega} P_a \, dm(a) = I$. Then $\mathcal{F}$ defined by

$$\mathcal{F}(E)\rho = \int_{E} P_a \rho P_a \, dm(a),$$

for a set $E \subseteq \Sigma$ and an operator $\rho \in \mathcal{F}(\mathcal{H})$, is an instrument. The corresponding observable is given by

$$x_\mathcal{F}(E)\rho = \int_{E} \text{tr}(P_a \rho) \, dm(a).$$

If all the projections in question are one-dimensional, then $\{P_a\}_{a \in \Omega}$ gives the coherent states resolution of unity. The model of a measurement presented above relates to normalized co-
herent states. However, in several cases it is convenient to reject the normalization assumption and require only that $\text{tr}(P_a)$ is constant with respect to $a$. We adopt this convention in Secs. V and VI. If $\mathcal{H}$ is finite dimensional, then the coherent states resolution of unity implies $\text{tr}(P_a) = \dim(\mathcal{H})/m(\Omega)$.

We call the density function $Q_\rho: \Omega \to \mathbb{R}^+$ given by the formula

$$Q_\rho(a) = \eta(P, a)$$

(5)

the generalized Husimi distribution corresponding to the state $\rho$. Schroeck$^{42}$ called this function the dequantization of the state $\rho$.

(F) The Davies instrument: simultaneous approximate momentum and position measurement

Let $\mathcal{H} = L^2(\mathbb{R}^3)$ and let $P, Q$ be the usual momentum and position operators. Let $\alpha \in \mathcal{H}$ be a unit vector in the domain of $P, Q$ such that $\langle Pa | a \rangle = \langle Qa | a \rangle = 0$. Let us define $\alpha_{yz} \in \mathcal{H}$ by $\alpha_{yz}(q) = \exp(i(z|q)) \alpha(q-y)$ for $q \in \mathbb{R}^3$ and $(y, z) \in \mathbb{R}^2$. Then $\alpha_{yz}$ are just the canonical coherent states. We can construct an instrument $\mathcal{F}$ as in example (E) taking $\Omega = \mathbb{R}^6$, $m$ the Lebesgue measure divided by $2\pi$ and $P_{yz} = |\alpha_{yz}\rangle \langle \alpha_{yz}|$ for $(y, z) \in \mathbb{R}^2$. Then the marginal observables of $x_\mathcal{F}$ defined by $x_\mathcal{F}(E) = x_{\mathbb{R} \times \mathbb{R}}(E)$ and $x_\mathcal{F}(F) = x_{\mathbb{R} \times \mathbb{R}}(R \times F)$, for $E, F \in \mathcal{B}(\mathbb{R})$, are the approximate position and momentum observables. The respective POV measures are given by the convolutions,$^{46,56}$ namely $A_1(E) = (\chi_F^* | \alpha|^2)(Q)$ and $A_2(F) = (\chi_F^* | \alpha|^2)(P)$ for $E, F \in \mathcal{B}(\mathbb{R})$, where $\chi_G$ denotes the characteristic function of a set $G$, for $G \in \mathcal{B}(\mathbb{R})$, and $\hat{\alpha}$ stands for the Fourier transform of the function $\alpha$. The map $(y, z) \to Q_\rho(y, z) = \langle \alpha_{yz}, p | \alpha_{yz} \rangle$ is the ordinary Husimi distribution$^{57,58}$ (or $Q$ function) of the state $\rho$. This function has become recognized as extremely useful for the analysis of quantum chaos.$^{59-64}$

Analogously, we can construct an instrument simultaneously measuring approximate momentum and position observables in the reduced phase space (cylinder or torus) or different spin components observables, where the phase space is the sphere. For this purpose, one should use the appropriate vector coherent states,$^{64,65}$ as it is demonstrated in Sec. VI.

(G) Classical mechanics: sharp and approximate measurements

Let $X, V, K$ be as in example (A) and let $(\Omega, \Sigma) = (X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel $\sigma$-algebra of $X$. Then $\mathcal{F}$ given by

$$\mathcal{F}(E) \mu(A) = \mu(A \cap E),$$

(6)

for $\mu \in \mathcal{V}$ and $A, E \in \Sigma$, is an instrument describing the sharp classical measurement. The corresponding observable has the form

$$x_\mathcal{F}(E) \mu = \mu(E),$$

(7)

for $E \in \Sigma$ and $\mu \in \mathcal{V}$. This measurement is nondemolition, i.e., $\mathcal{F}(\Omega) \mu = \mu$ for each $\mu \in \mathcal{V}$.

In order to show how one can model an approximate (or unsharp) classical measurement, we consider the following example. Let $X = \mathbb{R}^3$ and let $\mathcal{N}$ be a three-dimensional normal distribution with the zero mean. Then the instrument $\mathcal{F}$ given by

$$\mathcal{F}(E) \mu(A) = \int_A \int_E \mathcal{N}(y - u) \cdot \left( \int_X \mathcal{N}(y - z) d\mu(z) \right) dy du,$$

(8)

for $A, E \in \Sigma$ and $\mu \in \mathcal{V}$ with the observable

$$x_\mathcal{F}(E) \mu = \int_E (\mathcal{N}^* \mu)(y dy)$$

(9)

describe a measurement in question.
(H) Nuclear instruments

Cycon and Hellwig\(^\text{66}\) showed that many important instruments can be constructed in the following way (see also Refs. 67 and 68). Let \((V, K)\) be a state space, let \((\Omega, \Sigma)\) be a phase space, and let \(x\) be an observable. Let us take an \(x\)-measurable, \(x\)-essentially bounded state valued map \(\phi: \Omega \rightarrow V\). Then the formula

\[
\mathcal{I}(E) = \int_{E} \phi \, dx, \quad \text{for } E \in \Sigma,
\]

defines an instrument. Cycon and Hellwig called such an instrument nuclear. The integral in the formula is, so called, \((x)\)-integral. For the definition of \(x\)-measurability, \(x\)-essential boundedness, and \((x)\)-integral consult the Appendix in Ref. 66. Note that \(x^2 = x\) and \(\mathcal{I}(E)u = \int_E \phi dx (\cdot) u\) for all \(E \in \Sigma\) and \(u \in V\).

(I) Absolutely continuous nuclear instruments

Let \(V, K, \Omega, \Sigma,\) and \(x\) be as in the previous example. Let \(m\) be a measure on \((\Omega, \Sigma)\). Then \(x\) is absolutely continuous with respect to \(m\) (see Ref. 42), if there exists a measurable map \(f: \Omega \rightarrow \mathbb{R}\) such that \(\int_{\Omega} f \, dm = \tau\) and \(x(E) = \int_E f \, dm\) for each \(E \in \Sigma\). Then, the nuclear instrument from example (H) has a particularly simple form:

\[
\mathcal{I}(E)u = \int_{E} f(a)u \phi(a) \, dm(a),
\]

for \(E \in \Sigma\) and \(u \in V\). We call \(\mathcal{I}\) an absolutely continuous nuclear (ACN) instrument. Here, we can interpret \(f(a)u\) as the "transition probability density" from a state \(u\) to a measurement outcome \(a\) and \(\phi(a)\) as the state of the system after the measurement if the value \(a\) has been observed. Ozawa\(^\text{67,68}\) and Braccielli and Belavkin\(^\text{69}\) called \(\phi(\omega)\), where \(\omega \in \Omega\), an "a posteriori state."

We shall say that \(\mathcal{I}\) is informationally complete\(^\text{42}\) if \(u = w\), whenever \(f(b)u = f(b)w\) for almost every \(b\) (with respect to \(m\)).

The instruments considered in examples (D) and (E) are ACN instruments, if projections \(\{P_a\}_{a \in \Omega}\) are atomic, i.e., rank \(P_a = 1\) for all \(a \in \Omega\). This follows from the equality \(\text{tr}(P \rho)P = P \rho P\) valid for every atomic projection \(P\) and \(\rho \in \mathcal{F}(\mathcal{H})\). In this case \(f(a)\rho = \eta_{\rho}(a) = \text{tr}(P_a \rho)\) and \(\phi(a) = P_a\). Hence, a generalized coherent states instrument is informationally complete, if each density operator is uniquely determined by its generalized Husimi distribution. This is true, e.g., for canonical coherent states and spin coherent states.\(^{51}\) The instrument \(\mathcal{I}\) described in example (G) is also an ACN instrument with \(f(y)\mu = (\mathcal{J}^* \mu)(y)\) and \(\phi(y) = \mathcal{J}^\gamma\), for \(y \in \mathbb{R}^3\) and \(\mu \in V\), where \(\mathcal{J}^\gamma\) is the normal distribution \(\mathcal{N}\) translated by the vector \(y\).

(K) Transformed instrument

Let \((V, K)\) be a state space and let \((\Omega, \Sigma)\) be a phase space. Let us consider an isometric automorphism \(T: V \rightarrow V\) and an instrument \(\mathcal{I}: \Sigma \rightarrow \mathcal{G}\). The formula

\[
\mathcal{I}_T(E) : = T^{-1} \circ \mathcal{I}(E) \circ T,
\]

for \(E \in \Sigma\), defines a new instrument. We call it a transformed instrument.

C. Quantum stochastic processes

In this section we assume that \((V, K)\) is a state space, \(\Omega\) is a Polish space (i.e., metric, complete, and separable) and \(\Sigma\) its Borel \(\sigma\)-algebra.

Following Srinivas\(^\text{70}\) we can define a quantum stochastic process (QSP) as an arbitrary family of instruments \(\{\mathcal{I}_t\}_{t \in \mathcal{T}}\). The parameter "\(t\)" may be interpreted as a physical time. Let \(\mathcal{T} = \mathbb{Z}\) or \(\mathcal{T} = \mathbb{R}\), for discrete or continuous time domain, respectively. The finite-dimensional distributions of the process are measures \(\mu_{t_0, ..., t_{n-1}}^u\) defined on \((\Omega^n, \mathcal{B}(\Omega^n))\) as the natural extensions of the set functions given by
\[ \mu^u_{t_0,\ldots,t_n}(E_0 \times \cdots \times E_n) = \tau(\mathcal{I}_{t_{n-1}}(E_n) \circ \mathcal{I}_{t_{n-2}}(E_{n-2}) \circ \cdots \circ \mathcal{I}_{t_0}(E_0))u, \]

where \( n \in \mathbb{N}, t_0 < \cdots < t_{n-1} \), \( t_i \in \mathbb{T}, u \in V \), and \( E_0, \ldots, E_{n-1} \in \Sigma \). The meaning of these quantities is the following: \( \mu^u_{t_0,\ldots,t_{n-1}}(E_0 \times \cdots \times E_{n-1}) \) is the joint probability that the successive measurements of the system by the instruments \( \mathcal{I}_{t_0}, \ldots, \mathcal{I}_{t_{n-1}} \) at the moments \( t_0, \ldots, t_{n-1} \) give values in \( E_0, \ldots, E_{n-1} \), when the premeasurement state is \( u \).

Now, let us consider a system measured successively by an instrument \( \mathcal{I} \). Let us assume that between the measurements the system evolves and its evolution is described by a group \( \mathcal{G} = \{T_t\}_{t \in \mathbb{T}} \) of isometric automorphisms of \( V \). Then the complete evolution of the system can be described by the QSP \( \{\mathcal{I}_t\}_{t \in \mathbb{T}} \), where \( \mathcal{I}_t = T_{-t} \mathcal{I} T_t \) is a transformed instrument, i.e.,

\[ \mathcal{I}_t(E) = T_{-t} \mathcal{I}(E) T_t \]

for \( E \in \Sigma \). This way the time evolution of the system is included in the definition of the QSP (Heisenberg picture). We denote this process by \( \mathcal{G}(\mathcal{I}, \mathcal{J}) \). If \( \mathbb{T} = \mathbb{Z} \), then \( T_n = T^n \) for some isometric automorphism \( T: V \rightarrow V \) and \( n \in \mathbb{Z} \). In this case we shall use the notation \( \mathcal{G}(T, \mathcal{I}) \) instead of \( \mathcal{G}(\mathcal{I}, \mathcal{J}) \).

Various definitions of quantum Markov process have appeared in the literature. Instead of discussing all of them we propose the following "working" definition relevant to our purposes. We say that a QSP is Markovian if there exists a family of transition probabilities \( \{P_{t,s}\}_{t,s \in \mathbb{T}, s < t} \) such that

\[ \mu^u_{t_0,\ldots,t_n}(E_0 \times \cdots \times E_n) = \int_{E_0} \cdots \int_{E_n} P_{t_{n-1},t_n}(y_{n-1},dy_{n-1}) \cdots P_{t_0,t_1}(y_0,dy_0) \mu^u_{t_0}(dy_0), \]

for all \( t_0, \ldots, t_n \in \mathbb{T}, u \in V \) and \( E_0, \ldots, E_n \in \Sigma \). Let us recall that \( P: \Omega \times \Sigma \rightarrow \mathbb{R} \) is a transition probability if (i) \( P(\cdot, E) \) is a measurable map for each \( E \in \Sigma \), and (ii) \( P(x, \cdot) \) is a probability measure for every \( x \in \Omega \).

If the phase space \( (\Omega, \Sigma) \) is endowed with the Borel measure \( m \), \( P \) may have a form

\[ P(x,E) = \int_E p(x,y)dm(y). \]

where \( p: \Omega \times \Omega \rightarrow \mathbb{R}^+ \) is a transition function, i.e., a measurable map such that \( \int_\Omega p(x,y)dm(y) = 1 \) for every \( x \in \Omega \). It is worth pointing out that, in contrast to the theory of classical stochastic processes, the transitions probabilities of Markov QSP need not form a semigroup, i.e., the Smoluchowski–Chapman–Kolmogorov equation is not valid in general. We say that a quantum Markov process is homogeneous if the transition probabilities \( P_{t,s} \) depend only on the difference \( t-s \). In this case we shall write the transition probabilities \( P_{t,s} \) as \( P_{t-s} \), and the transition functions \( p_{t,s} \) as \( p_{t,s} \).

In the sequel we shall use the following simple proposition (for the proof see Appendix A).

**Proposition 1:** Let \( \mathcal{I} \) be an absolutely continuous nuclear instrument considered in example (J) and let \( \mathcal{G} = \{T_t\}_{t \in \mathbb{T}} \) be a semigroup of isometric automorphisms of the state space \( V \). Then the process \( \mathcal{G}(\mathcal{I}, \mathcal{J}) \) is a homogeneous quantum Markov process with the transition functions given by

\[ p_t(a,b) = f(b)(T_t(\phi(a))) \]

for \( t \in \mathbb{T} \) and \( a,b \in \Omega \).

For simplicity, we shall consider only discrete-time QSP in the sequel. Note, however, that the results can be easily generalized to the continuous-time case.
The following examples of quantum Markov processes seem to be specially interesting.

(L) Sharp measurement of evolving classical system

Let $\mathcal{I}$ be the instrument described in example (G) and let $\Theta: X \rightarrow X$ be a measurable map. Then $\Theta$ generates the automorphism $T_\Theta: V \rightarrow V$ by $T_\Theta(\mu)(A) = \mu(\Theta^{-1}(A))$ for $\mu \in V$ and $A \in B(X)$. It is easy to show that the process $\mathcal{E}(T_\Theta, \mathcal{I})$ is a homogeneous quantum Markov process and its transition probability is given by $P(x, E) = \chi_E(\Theta x)$ for $x \in X$ and $E \in B(X)$.

(M) Measurement of evolving quantum system

Let $\mathcal{I}$ be the instrument discussed in example (D) or (E) such that each $P_a$ is atomic, for $a \in \Omega$, i.e., $P_a = |\alpha_a\rangle\langle \alpha_a|$ for some $|\alpha_a\rangle \in \mathcal{H}$. From now on we consider a modification of this instrument putting aside the normalization condition for CS and allowing $(\alpha_a| \alpha_a)$ to be equal to an arbitrary positive constant $D$. Let us assume that the system evolves according to the Schrödinger equation and its evolution is given by $T_U: \mathcal{H}(\mathcal{H}) \rightarrow \mathcal{H}(\mathcal{H})$, where $T_U(\rho) = U^{-1} \rho U$, for some unitary operator $U$. It follows from Proposition 1 that $\mathcal{E}(T_U, \mathcal{I})$ is a homogeneous quantum Markov process with the transition function given by

$$p(a, b) = \text{tr}(P_b U^{-1} P_a U)/D$$

for $a, b \in \Omega$. Consequently, we obtain

$$p(a, b) = |(\alpha_a| U| \alpha_a)|^2/D.$$  

Special cases of formula (19) were considered by several authors. For instance, Beck and Graudenz\textsuperscript{30} derived it for the Lüders-von Neumann instrument. Dittrich and Graham\textsuperscript{72} also used this formula in their analysis of the repeated precise measurement of the action variable in the quantized standard map. In the latter paper $\Omega = \mathbb{Z}$ and $\{\alpha_a\}_{a \in \mathbb{Z}}$ were the eigenstates of the angular momentum of the kicked rotator.

Each discrete ($\Sigma = \mathbb{Z}$) homogeneous quantum Markov process with one-step transition probability $P$ induces the operator $M^*$ in the space of all probability measures on $(\Omega, \Sigma)$ given by

$$M^*(\mu)(E) = \int P(a, E) d\mu(a),$$

where $\mu$ is a probability measure on $(\Omega, \Sigma)$ and $E \in \Sigma$. The operator $M^*$ assigns to the actual distribution of the measurement results the anticipated distribution after the next measurement. We say that a state $\mu$ is stationary if

$$M^*(\mu_0) = \mu_0.$$  

If $P$ is given by the transition function $p$, then instead of the operator $M^*$ we can use the Markov operator with stochastic kernel\textsuperscript{73} $M: L^1(\Omega, m) \rightarrow L^1(\Omega, m)$ to describe the evolution of the probability densities of the measurement outcomes. This operator is given by

$$(Mg)(b) = \int g(a) p(a, b) dm(a),$$

for $g \in L^1(\Omega, m)$. The following proposition provides a simple criterion for stationarity. We prove it in Appendix B.

**Proposition 2:** (A) Let $\mathcal{I}$ be an absolutely continuous nuclear instrument [see example (J)] and let $T: V \rightarrow V$ be an isometric automorphism. Then the state $\mu$ is stationary for $\mathcal{E}(T, \mathcal{I})$ if $T(\mathcal{I})\mu = \mu$. If $\mathcal{I}$ is informationally complete, then the converse implication also holds.

(B) Let $\mathcal{I}$ be a generalized coherent states instrument from example (M) and let $T = T_U$ for some unitary operator $U$. If $U^{-1}(\mathcal{I})\rho U = \rho$, then the state $\rho$ is stationary for $\mathcal{E}(T, \mathcal{I})$. In this
case, the generalized Husimi distribution $Q_{\rho}$ satisfies the equation $M(Q_{\rho}) = Q_{\rho}$. If $\dim \mathcal{H} = N$, then $\rho = 1/N \cdot 1$ is stationary and $Q_{\rho} = 1/m(\Omega) = D / N$, where the constant $D$ is equal to the squared norm of a coherent state.

(C) Let $\mathcal{F}$ be the instrument describing the sharp classical measurement [see example (L)], and let $T = T_{\Theta}$ for some measurable classical map $\Theta : X \rightarrow X$. Then $\mu$ is stationary for $\mathcal{S}(T, \mathcal{F})$ if and only if $\mu$ is a $\Theta$-invariant measure.

III. ENTROPY—GENERAL DEFINITION

There exist various concepts of entropy, in both the classical and quantum cases. Some of them concern states of a system and these go back to Boltzmann and Gibbs. The others connect the notion with the dynamics of a system. This approach may be traced back to Shannon and Kolmogorov. In this case we say sometimes about an entropy per unit time. References 29, 31, and 38 contain the detailed review of the subject.

Our approach to quantum entropy is the following. Using Kiefer’s concept of entropy with respect to partition, we define entropy for a triple: a quantum stochastic process, a partition of a phase space, and an initial state. Considering a special case of this definition, we define entropy of a quadruple: a quantum map, an instrument, a partition of a phase space, and an initial state. This way of introducing entropy is similar to the one proposed by Pechukas and, independently, by Beck and Graudenz. In fact, our notion of entropy contains theirs as a special case. Applying our definition to a classical system one obtains the Kolmogorov–Sinai entropy. In this sense our approach is the generalization of the classical one.

Let $(V, K)$ be a state space, let $u \in K$, and let $\tau(u) = 1$. Let $\Omega$ be a Polish space, let $\Sigma$ be its Borel $\sigma$-algebra, and let $\mathcal{A} = \{E_1, \ldots, E_k\}$ be a measurable partition of $\Omega$.

Let $\mathcal{C} = \{\mathcal{P}_t\}_{t \in N}$ be a QSP. We define the entropy of $\mathcal{C}$ in the initial state $u$ with respect to the partition $\mathcal{A}$ as the limit

$$h(\mathcal{C}, \mathcal{A}, u) := \lim_{n \to \infty} \sup \frac{1}{n} c_n,$$

where

$$c_n = - \sum_{i_0, \ldots, i_{n-1} = 1}^k \mu_{i_0, \ldots, i_{n-1}}^u(F_{i_0} \times \cdots \times E_{i_{n-1}}) \ln(\mu_{i_0, \ldots, i_{n-1}}^u(E_{i_0} \times \cdots \times E_{i_{n-1}})).$$

(23)

where the convention $0 \ln(0) = 0$ is adopted. This quantity measures the degree of randomness of the sequences of the measurement outcomes. Our definition of entropy mimics that of Shannon and Kolmogorov. It is also possible to generalize the notion of Rényi order-$\alpha$ entropy in a similar way (see Appendix C).

For quantum Markov processes the existence of the limit in the definition of entropy (23) can be derived from the classical theorems on entropy (see Ref. 76, Theorem 4.10 and Ref. 43, p. 37). Namely, the following result holds.

Proposition 3: Let $\mathcal{C}$ be a Markov quantum process and let $u$ be a stationary state. Then the sequence $c_n/n$ in the formula (23) decreases with $n$ to $h(\mathcal{C}, \mathcal{A}, u)$.

Now let $\mathcal{F}$ be an instrument, let $T : V \to V$ be an isometric automorphism such that $\mathcal{C}(T, \mathcal{F})$ is a quantum Markov process, and let $\mathcal{A} = \{E_1, \ldots, E_k\}$ be a partition of the phase space $\Omega$. Then we define the (quantum) entropy of the map $T$ with respect to the instrument $\mathcal{F}$ and the partition $\mathcal{A}$ in the state $u$ as

$$H(T, \mathcal{F}, \mathcal{A}, u) = h(\mathcal{C}(T, \mathcal{F}), \mathcal{A}, u),$$

(24)

where $\mathcal{C}(T, \mathcal{F})$ is a QSP described in Sec. II C, with finite-dimensional distributions given by
for \( n \in \mathbb{N}, i_0, \ldots, i_{n-1} = 1, \ldots, k \).

In the quantum case the partition \( \mathcal{B} \) may be given the following interpretation. Its elements \( E_i, i = 1, \ldots, k \), can be treated as symbols used to encode the results of the successive double (multiple) measurements, which specify points of the classical phase space. This reminds the description of the classical system by the symbolic dynamics.

As it has already been mentioned, there are two independent sources of the randomness in our model. The first is the process of measurement itself described by the instrument \( \mathcal{F} \). The second is the underlying dynamics of the system described by the isometric automorphism \( T \). Clearly, both of them can influence the value of the quantum entropy. To measure these two kinds of randomness separately, we introduce the following quantities:

\[
H_{\text{mes}}(\mathcal{F}, \mathcal{B}, u) := H(\mathcal{I}, \mathcal{F}, \mathcal{B}, u),
\]

\[
H_{\text{dyn}}(T, \mathcal{F}, \mathcal{B}, u) := H(T, \mathcal{F}, \mathcal{B}, u) - H_{\text{mes}}(\mathcal{F}, \mathcal{B}, u),
\]

where \( I \) is the identity automorphism on \( V \), and call them, measurement quantum entropy and dynamical quantum entropy, respectively.

The following facts are easy to show.

**Proposition 4:** (A) Let \( \mathcal{F}, T \) be as in example (L), let \( \mathcal{B} \) be a finite measurable partition of \( X \), and let \( \mu \) be an invariant probability measure [i.e., \( \mu(X) = 1 \)] for \( \Theta \). Then \( H(T, \mathcal{F}, \mathcal{B}, \mu) \) is equal to the Kolmogorov–Sinai entropy of \( \Theta \) with respect to \( \mathcal{B} \) (see Ref. 76, Definition 4.9). We shall denote this quantity by \( H(\Theta, \mathcal{B}) \).

(B) Let \( \mathcal{F} \) be the Lüders–von Neumann instrument, as in example (D), let \( \mathcal{B} = \{i\} : i = 1, \ldots, N \), and let \( T : \mathcal{F}(\mathcal{B}) \to \mathcal{F}(\mathcal{B}) \) be given by \( T(\rho) = U^{-1} \rho U \) for some \( U \), unitary operator on \( \mathcal{H} \). Then \( H(T, \mathcal{F}, \mathcal{B}, (1/N) I) \) is greater than or equal to the Pechukas–Beck–Graudenz entropy for constant measuring.\(^{18,20,30}\) If \( P_i = |\Psi_i\rangle\langle\Psi_i| \) for \( \Psi_i \in \mathcal{H}, i = 1, \ldots, N \), then

\[
H \left( T, \mathcal{F}, \mathcal{B}, \frac{1}{N} I \right) = -\frac{1}{N} \sum_{i,j=1}^{N} a_{ij} \ln a_{ij},
\]

where \( a_{ij} = |\langle \Psi_i | U | \Psi_j \rangle|^2 \), for \( i, j = 1, \ldots, N \).

(C) Let \( A \) be a C*-algebra, let \( V \) be its state space defined as in example (C), let \( \Theta \) be an automorphism of \( A \) and let \( \omega \) be a state such that \( \omega \Theta = \omega \). Moreover, let \( X = \{x_1, \ldots, x_k \} \subset A \) be a finite operational partition of unity. Let us define an instrument \( \mathcal{F} \) and an isometric automorphism \( T : V \to V \) by the formulas \( \mathcal{F}(\{i\})(v)(a) = v(x_i^* a x_i) \) [then the corresponding observable is given by \( x_i(\{i\})(v) = v(x_i^* x_i) \) and \( T(v)(a) = v(\Theta a) \) for \( i = 1, \ldots, k \), \( v \in V \) and \( a \in A \).] Let us denote \( \mathcal{B} = \{\{i\} : i = 1, \ldots, k \} \). Then \( H(T, \mathcal{F}, \mathcal{B}, \omega) \) is equal to the dynamical entropy \( H_{\text{dyn}}(\omega, \Theta)(X) \) discussed by Alicki and Fannes.\(^{39}\)

Thus our approach covers the classical KS-entropy, the Pechukas–Beck–Graudenz entropy of constantly measured observable with discrete spectrum, and, in a sense, a recent concept of Alicki and Fannes. Note that the measurement entropy vanishes in the first two cases and, consequently, \( H = H_{\text{dyn}} \).

**Remark 1:** The first attempt to give an exact quantum analogue of the classical KS-entropy was made by Kosloff and Rice.\(^{18,19}\) They remarked that if a quantum entropy was linear in the time between measurements, then it would be equal to zero for every quantum system with a discrete spectrum. However, the entropy introduced in Refs. 18 and 19 does not necessarily satisfy this requirement. For instance, if the unitary evolution operator is equal to identity, then the entropy introduced by Kosloff and Rice is just Ingarden–Urbanik measurement entropy.\(^{15}\) The similar approach was proposed independently by Partovi.\(^{25}\) Both concepts were based on a sam-
pling measurement procedure, i.e., on the assumption that the motion of the system is not disturbed by the measurement process. In contrast, we assume that the state of the system changes after each act of measurement.

Remark 2: Kosloff and Rice\textsuperscript{18} also discussed briefly an alternate definition, which was studied in details by Pechukas\textsuperscript{20} later on. This definition was rediscovered recently by Beck and Graudenz\textsuperscript{30} (see Proposition 4(B)). It was also discussed implicitly by Gaspard\textsuperscript{31,35} and Lindblad.\textsuperscript{37} They investigated the limit behavior of n time correlation functions, whose diagonal elements can be identified with the finite-dimensional distributions of the QSP describing the evolution of a quantum system successively measured by the Lüders–von Neumann instrument. It is not clear whether the Pechukas–Beck–Graudenz entropy has the correct semiclassical limit. However, it seems to be doubtful, as calculations in Ref. 20, Sect. V, indicate that this entropy may be in some situations, paradoxically, lower in the chaotic regime than in the regular one (see also discussions in Refs. 21 and 77). Besides, Gaspard’s analysis\textsuperscript{31} of the correlation functions for the quantum baker map suggests that Pechukas–Beck–Graudenz entropy may be even equal to zero in the classically chaotic case.

Remark 3: Helton and Tabor\textsuperscript{21} defined quantal and classical entropies, which can also be put into our general framework. In fact, their classical entropy corresponds to a classical unsharp measurement and the quantal entropy to a modification of the Lüders–von Neumann measurement (see also Appendix D). Helton and Tabor claimed that their quantal entropy tends to the classical entropy in the limit as $\hbar \to 0$, but they did not give a proof.

Remark 4: Comes, Størmer, Namhoffer, and Thirring\textsuperscript{16,23,26,32,38} gave another definition of the dynamical entropy of a quantum system based on the theory of $C^*$-algebras. This definition may be also interpreted in terms of a measurement process.\textsuperscript{34,35} However, only Lüders–von Neumann type of measurement is involved in the Connes–Namhoffer–Thirring construction and the method of subtracting the randomness connected with the measurement to obtain the dynamical component of quantum entropy is different than ours. This algebraic approach, though generalizes the notion of classical KS-entropy,\textsuperscript{35} results in entropy that is equal to zero for a finite quantum system and hence it cannot be used to detect quantum chaos in such a case. Another definition of quantum entropy based on the theory of $C^*$-algebras has recently been stated by Alicki and Fannes\textsuperscript{39} (see Proposition 4(C)).

IV. ENTROPY—COHERENT STATES INSTRUMENTS

In this section we would like to give an explicit formula for quantum entropy in the case that is probably the most interesting from the “quantum chaotic” point of view. Let the evolution of a quantum system be given by a quantum map $T_U : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$, defined by $T_U(\rho) = U^{-1} \rho U$, for $\rho \in \mathcal{F}(\mathcal{H})$, where $U$ is a unitary operator on $\mathcal{H}$. Moreover, let us assume that the system is measured at equal time intervals by a generalized coherent states instrument $\mathcal{F}$ [see example (M)]. Let $\mathcal{A} = \{E_1, \ldots, E_k\}$ be a partition of the phase space $\Omega$ and let $\rho$ be a stationary state for $U$ and $\mathcal{F}$ [see Proposition 2(B)]. Let us put $H(U, \mathcal{F}, \mathcal{A}, \rho) = H(T_U, \mathcal{F}, \mathcal{A}, \rho)$. To shorten notation, we write $|\alpha\rangle$ for a coherent state $|\alpha\rangle$, where $a \in \Omega$. Let $\langle a | a \rangle = D > 0$.

Applying formulas (13), (15), (16), and (19) one may compute the appropriate joint probabilities of successive measurements. Probability that the result of the measurement at $t=0$ will be encoded by symbol $E_{i_0}$ is

$$P_{i_0}^{CS} = \int_{E_{i_0}} \langle a_0 | \rho | a_0 \rangle dm(a_0).$$

(27)

where the index $i_0 = 1, \ldots, k$. After the second measurement at $t=1$ the quantum path is encoded by two symbols. Sequence $\{E_{i_0}, E_{i_1}\}$ occurs with the probability given by the double integral

$$\int_{E_{i_0}} \int_{E_{i_1}} \langle a_0 | \rho | a_0 \rangle \langle a_1 | \rho | a_1 \rangle dm(a_0) dm(a_1).$$

(28)
\[ P_{i_0, \ldots, i_{n-1}}^\text{CS} = \int_{E_{i_0}} \cdots \int_{E_{i_{n-1}}} \langle a_0 | \rho | a_0 \rangle \frac{|\langle a_1 | U | a_0 \rangle|^2}{D} \, dm(a_1) dm(a_0). \] (28)

In general, the \( n \)-symbols sequence \( \{E_{i_0}, \ldots, E_{i_{n-1}}\} \) is found with the probability

\[ P_{i_0, \ldots, i_{n-1}}^\text{CS} = \int_{E_{i_0}} \cdots \int_{E_{i_{n-1}}} \langle a_0 | \rho | a_0 \rangle \prod_{i=1}^{n-1} \left[ \frac{ka_i | U | a_{i-1} \rangle |^2}{D} \right] \, dm(a_{i-1}) \cdots dm(a_0), \] (29)

where all indices \( i_0, \ldots, i_{n-1} \) run from 1 to \( k \). This formula is based on the assumption that after each intermediate approximate measurement of the conjugated observables the quantum system evolves as an appropriate coherent state. For example, the probability \( P_{012}^\text{CS} \) of the transition \( E_0 \to E_1 \to E_2 \) is summed over all coherent states \( |a_i\rangle \) from the cell \( E_i \). Note that for standard examples of coherent states the quantities \( P_{CS} \) tend in the limit \( \hbar \to 0 \) to the corresponding classical probabilities, which enter the definition of Kolmogorov-Sinai entropy. Moreover, for any finite number of measurements \( n \) the sum of probabilities carried over all possible \( n \)-symbol sequences is equal to the unity

\[ \sum_{i_0, \ldots, i_{n-1}=1}^{k} P_{i_0, \ldots, i_{n-1}}^\text{CS} = 1. \] (30)

This follows from definition (13), but may be verified directly by summing the domains of integrations in Eq. (29) and applying \( (n-1) \) times the identity resolution \( \int \langle a | a \rangle dm(a) = 1 \). Condition (30) allows us to interpret the quantities \( P_{CS} \) as quantum analogues of classical probabilities.

Applying (29) to the general definition of entropy (23) and (24) we obtain

\[ H(U, \mathcal{I}, \mathcal{M}, \rho) = \lim_{n \to \infty} \frac{1}{n} c_n, \]

where

\[ c_n := - \sum_{i_0, \ldots, i_{n-1}=1}^{k} P_{i_0, \ldots, i_{n-1}}^\text{CS} \ln P_{i_0, \ldots, i_{n-1}}^\text{CS}, \] (31)

and the convention \( 0 \ln(0) = 0 \) is adopted.

We shall call this quantity \emph{coherent states quantum entropy}, or, briefly, CS-quantum entropy. Let us recall that measurement and dynamical components of CS-quantum entropy are

\[ H_{\text{mes}}(\mathcal{I}, \mathcal{M}, \rho) = H(I, \mathcal{I}, \mathcal{M}, \rho) \] (32)

and

\[ H_{\text{dyn}}(U, \mathcal{I}, \mathcal{M}, \rho) = H(U, \mathcal{I}, \mathcal{M}, \rho) - H_{\text{mes}}(\mathcal{I}, \mathcal{M}, \rho). \] (33)

In contrast to Kolmogorov-Sinai entropy or Pechukas-Beck-Graudenz entropy, measurement CS-quantum entropy might be greater than 0 (see Sec. VI), due to nonzero overlap of different coherent states. However, it seems reasonable to suspect that CS-quantum entropy attains its minimum (with a coherent states instrument and a partition fixed) for the identity evolution operator. Therefore we would like to pose the following conjecture.

**Conjecture 1:** \( H_{\text{dyn}}(U, \mathcal{I}, \mathcal{M}, \rho) \geq 0 \).
Strict positivity of dynamical CS-quantum entropy may be proposed as a quantitative criterion of quantum chaos. This, however, requires more detailed studies, as the character of the dependence of this quantity on the time between two successive measurements and on the partition of the phase space is not completely clear (see also Remark 6 below). Note that the significance of fuzzy measurements in describing and explaining quantum chaos has already been indicated by Peres.78

Remark 5: Definition (29) of joint quantum probabilities is related to the concept of quantum histories introduced by Griffiths, Gell-Mann, and Hartle, and Omnès.79–83 A brief discussion of the definition of quantum probabilities and quantum entropy in this context is given in Appendix D.

Remark 6: We connect the notion of entropy with a pair: an instrument and a partition of the phase space. However, one can ask what happens if we increase the number k, taking partition finer and finer. Then, since the function x ln(x) is convex, the entropy increases. It is well known that for classical chaotic systems the sequence $c_n$ stabilizes for sufficiently fine scale (so-called generating partition) and the limit is achieved (see, e.g., Ref. 76, Theorem 4.20). This limit is equal to the Kolmogorov–Sinai (KS) entropy of the system. In contrast, the CS-quantum entropy tends to the infinity, if the diameter of the partition tends to 0, which can be easily derived from Ref. 43, Theorem 2.4. The following question seems to be interesting: what happens to the dynamical CS-quantum entropy when the diameter of partition tends to 0? We do not know the answer. Note that in this case the dynamical component of CS-quantum entropy is a difference of two infinite quantities.

Remark 7: In a series of papers Weigert84–87 proposed a model of quantum chaos utilizing the notion of quantum-nondemolition measurement. He investigated a quantized version of Arnold’s cat map. The unitary transformation $U$ describing the dynamics of the system acts in such a way that the transition function (19) is given by $p(a,b) = |\langle \beta_b | U | \beta_a \rangle|^2/D = |\langle \beta_b | \delta_{\Theta(a)} | \beta_a \rangle|^2/D = \delta_{\Theta(a)}(b)$, where $a,b$ are points on the two-dimensional torus $T^2$, $\Theta$ is the cat map on $T^2$, and $\delta$ stands for $\delta$-Dirac distribution. Then, the formal calculation yields the quantum entropy of the system equal to the Kolmogorov–Sinai entropy of the cat map, and hence positive. However, $|\beta_a \rangle : a \in T^2$ do not constitute a set of coherent states, but are the eigenstates of the position operator labeled by the elements of $T^2$. In fact they are not states (elements of a Hilbert space) in the strict sense of the word, but rather generalized states. The transition function $p$ is discontinuous and it is not clear whether such a generalization relates to any physically meaningful measurement process. It turns out that the kind of measurement Weigert considered is only an idealization of a real measurement of position variable. Actually only approximate measurements of position are possible and “eigenstates” of position operator should be replaced by an appropriate family of squeezed coherent states (see Ref. 56 for details). Hence Weigert’s model cannot be put literally into our general framework.

In the next section we try to compare the KS-entropy of a classical dynamical system and the coherent states quantum entropy of its quantized version.

V. CORRESPONDENCE PRINCIPLE FOR ENTROPY

Let the classical phase space $X$ be a compact Hausdorff space and let the classical map $\Theta: X \rightarrow X$ be a homeomorphism. Let $m$ be a $\Theta$-invariant probability measure on $X$. The quantization procedure for this system consists of two steps. First, we have to define the kinematics of the quantized system, i.e., its state space $\mathcal{H}$, and then the dynamics, i.e., the unitary evolution operator on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, which depends on the scale parameter $\hbar$, the relative Planck constant (see, e.g., Ref. 88). As the underlying phase space $X$ is compact, so the Hilbert space $\mathcal{H}$ is finite dimensional and the operator $U$ may be represented by a unitary matrix with the dimension $N$ proportional to $1/\hbar$.

Applying the quantization procedure we obtain the family of Hilbert spaces $\{\mathcal{H}_\hbar\}_{\hbar \in S}$ and the family of unitary operators $\{U_\hbar\}_{\hbar \in S}$, where $S$ is a subset of $\mathbb{R}^+$ with a cluster point at 0. Let us
assume that the family of coherent states \( \{ | \alpha^\hbar_x \rangle : x \in X \} \) is given for each \( \hbar \in S \). This means that we have the family of continuous maps \( \alpha^\hbar : X \to \mathcal{A} \) given by \( \alpha^\hbar(x) = | \alpha^\hbar_x \rangle \), for \( x \in X \), which fulfills condition

\[
\int_X | \alpha^\hbar_x \rangle \langle \alpha^\hbar_x | dm(x) = 1, \tag{34}
\]

for each \( x \in X \), with the norm \( \| \alpha^\hbar_x \| = \sqrt{D_\hbar} \) constant with respect to \( x \) [see example (M)]. The constant \( D_\hbar \) is equal to \( \dim (\mathcal{A}_\hbar) \), as \( m(X) = 1 \).

Now we introduce the following basic definition.

We say that the quantization procedure \( \{ (\mathcal{A}_\hbar, \{ U_\hbar \}_{\hbar \in S}) \} \) is regular with respect to the family of coherent states \( \{ | \alpha^\hbar_x \rangle : x \in X \}_{\hbar \in S} \) if

\[
\lim_{\hbar \to 0} \sup_{x \in X} \int_X G(b) \left( \frac{\langle \alpha^\hbar_x | U_\hbar | \alpha^\hbar_x \rangle^2}{D_\hbar} \right) dm(b) - g(\Theta a) = 0, \tag{35}
\]

for every continuous function \( g : X \to \mathbb{R} \).

The assumption of regularity has the following physical interpretation. The members of the difference in (35) are the quantum expectation value of \( g \) in the state \( U_\hbar | \alpha^\hbar_x \rangle \) and the classical expectation value of \( g \) in the state \( \delta_{\Theta a} \), respectively. Condition (35) is equivalent to the following one (see Appendix E).

For each sufficiently small \( \delta > 0 \)

\[
\lim_{\hbar \to 0} \inf_{a \in X} \int_{B(\Theta a, \delta)} \left( \frac{\langle \alpha^\hbar_x | U_\hbar | \alpha^\hbar_x \rangle^2}{D_\hbar} \right) dm(b) = 1. \tag{36}
\]

Simple interpretation of the above condition is shown in Fig. 1.

In the sequel we shall assume that the quantization procedure is regular.

Let \( \mathcal{A} = \{ E_1, \ldots, E_k \} \) be a finite measurable partition of \( X \) such that \( m(\partial \mathcal{A}) = m \left( \bigcup_{i=1}^k \partial(E_i) \right) = 0 \). Let \( \mathcal{A}_\hbar \) be the coherent states instrument connected with the family of coherent states \( \{ | \alpha^\hbar_a \rangle : a \in X \} \). We assume that \( \rho_\hbar \) is a stationary state for the combined evolution given by the operator \( U_\hbar \) and the generalized coherent states instrument \( \mathcal{A}_\hbar \), such that the generalized Husimi distribution \( Q_{\rho_\hbar} = 1 \), e.g., \( \rho_\hbar = (1/D_\hbar) \cdot I \) [if \( \mathcal{A}_\hbar \) is informationally complete this is the only possible choice—see Proposition 2(A) and (B)].

We can now formulate our main result.

**Correspondence principle for quantum entropy:**

\[ Q \]

\[ \Theta \]

\[ a \]

\[ B(\Theta a, \delta) \]

\[ \mathcal{A} \]

\[ \mathcal{A}_\hbar \]

\[ U_\hbar \]

\[ Q_{| \alpha^\hbar_a \rangle} \]

\[ Q_{U_\hbar | \alpha^\hbar_a \rangle} \]

FIG. 1. Regular quantization. Procedure linking a family of quantum maps \( \{ U_\hbar \} \) with a classical map \( \Theta \) is regular, if the integral of the Husimi distribution \( Q \) of the transformed coherent state \( U_\hbar | \alpha^\hbar_a \rangle \) taken over a ball of an arbitrary small radius \( \delta \) localized at the classical image \( \Theta a \) of an element \( a \) of the phase space \( X \), tends in the semiclassical limit to the unity, uniformly with respect to \( a \).
\[ \limsup_{\hbar \to 0} H(U_{\hbar}, \mathcal{F}, \rho_{\hbar}) \leq H(\Theta) \leq H(\Theta), \]

(37)

where \( H(\Theta) \) denotes the KS-entropy of the classical map \( \Theta \).

To prove the above result it suffices to use the appropriate theorems on entropy of random perturbations of dynamical systems. See Appendix F for details. The fact that the CS-quantum entropy is bounded in the limit as \( \hbar \to 0 \) by KS-entropy does not contradict Remark 6, as the limits \( k \to \infty \) and \( \hbar \to 0 \) do not commute.

Moreover, we believe that the following conjecture, analogous to the Kiefer’s “entropy via random perturbations” theorem, is true for a broad class of classical maps and their quantizations.

**Conjecture 2:** If \( X \) is a smooth compact manifold and \( \Theta \) is a \( C^2 \) hyperbolic transitive diffeomorphism of \( X \), then there exists \( \epsilon > 0 \) such that \( \text{diam}(\mathcal{A}) \leq \epsilon \) implies

\[ \lim_{\hbar \to 0} H(U_{\hbar}, \mathcal{F}, \rho_{\hbar}) = H(\Theta), \]

(38)

where \( \text{diam}(\mathcal{A}) = \max(\text{diam}(E_1), \ldots, \text{diam}(E_k)) \).

**Remark 8:** In condition (36) one can use an arbitrary metric on the compact space \( X \). Note, however, that a family of coherent states itself generates a Riemannian metric on \( X \). See Refs. 50 and 54 for the general case and Ref. 89 for the SU(\( N \)) coherent states. In the latter paper the Riemannian metric is given by the logarithm of the modulus of the coherent states overlap.

**VI. EXAMPLE—SU(2) COHERENT STATES**

In order to illustrate the definition of quantum entropy we discuss classical systems with the phase space homeomorphic to the two-dimensional sphere \( S^2 \). Due to compactness of the phase space, the Hilbert space \( \mathcal{H}_N \) describing the corresponding quantum dynamics is finite dimensional. Consider three components \( \{J_x, J_y, J_z\} \) of the angular momentum operator \( J \). These operators are related to the infinitesimal rotations along three orthogonal axis \( \{x, y, z\} \) in \( \mathbb{R}^3 \) and fulfill the standard commutation relation

\[ [J_k, J_l] = i\epsilon_{klm}J_m, \]

(39)

where \( k, l, m = x, y, z \) and \( \epsilon_{klm} \) represents the antisymmetric tensor. Operators \( J_x = J_x \pm iJ_y \) and \( J_z \) are generators of the SU(2) group. Eigenvalue \( j(j+1) \) of the Casimir operator \( J^2 = J_x^2 + J_y^2 + J_z^2 \) determines the dimension \( N = 2j + 1 \) of the representation of the group. Since the semiclassical limit is obtained by letting \( j \to \infty \), an integer or half-integer quantum number \( j \) might be considered as proportional to \( \hbar^{-1} \). Common eigenstates \( |j, m\rangle, m = -j, \ldots, j \) of the operators \( J^2 \) and \( J_z \) form an orthonormal basis in \( \mathcal{H}_N \). It is convenient to choose as a reference state the state \( |j, j\rangle \) (of maximal weight) and to define the SU(2) coherent states by

\[ |j, \gamma\rangle = \sqrt{2j+1} \exp[\gamma J_z] |j, j\rangle, \]

(40)

where \( \gamma \) is an arbitrary complex number. These coherent states (CS) can be expanded in the eigenbasis of \( J^2 \) and \( J_z \) as

\[ |j, \gamma\rangle = \sqrt{2j+1} \sum_{m=-j}^{m=j} \gamma^{j-m}(1+\gamma \bar{\gamma})^{-1/2} \left( j-m \right)^{1/2} |j, m\rangle. \]

(41)

A stereographical projection \( \gamma = \tan(\theta/2)\exp(i\varphi) \) links a coherent state \( |j, \gamma\rangle = |j, \theta, \varphi\rangle \) to a point \( (\theta, \varphi) \) on the sphere \( S^2 \). Note that \( S^2 \) is isomorphic to the coset space SU(2)/U(1), where U(1) is the maximal stability subgroup of SU(2) with respect to the state \( |j, j\rangle \), i.e., the subgroup of all
elements of SU(2) that leave $|j,j\rangle$ invariant up to a phase factor. Hence the above construction may be treated as a particular case of the general construction of group-theoretic coherent states\cite{52,54,90} and, in consequence, as a particular case of our examples (E) and (M). The appropriate instrument describes unsharp simultaneous measurement of different spin components.\cite{65,91} We introduce the factor $\sqrt{2j+1}$ in formulas (40) and (41) in order to ensure the coherent states identity resolution in the form

$$\int_{S^2}|j,\theta,\varphi\rangle\langle j,\theta,\varphi|d\mu(\theta,\varphi)=I,$$

where the Riemannian measure $\mu$ on $S^2$ is given by $d\mu=\sin\theta d\theta d\varphi/4\pi$ and so does not depend on the quantum number $j$. This enables the respective Husimi distribution of the coherent states to tend to the Dirac $\delta$-function as $\hbar\to 0$ ($j\to\infty$).

Expectation values of the components of $J$ are

$$\langle j,\theta,\varphi|J|j,\theta,\varphi\rangle=j(2j+1)(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$$

and justify the interpretation of CS as vector states oriented along the direction defined by a point on the sphere.\cite{33,54,90} Similar formulas written in the complex number representation

$$\langle j,\gamma|J_\pm|j,\gamma\rangle=2j(2j+1)\gamma/(1+\gamma^2)$$

and

$$\langle j,\gamma|J_z|j,\gamma\rangle=2j(2j+1)\gamma/(1+\gamma^2)$$

allow us to obtain the relations

$$\gamma=\frac{\langle j,\gamma|J_\pm|j,\gamma\rangle}{j(2j+1)+\langle j,\gamma|J_\pm|j,\gamma\rangle}.$$  

Consider now a unitary operator $U$ governing the quantum dynamics in a $(2j+1)$-dimensional Hilbert space. In the analogy to the identity (46) we define a family of classical maps by

$$\Theta_j(\gamma)=\frac{\langle j,\gamma|U^\dagger J_\pm U|j,\gamma\rangle}{j(2j+1)+\langle j,\gamma|U^\dagger J_\pm U|j,\gamma\rangle}.$$  

Letting $j\to\infty$ we may obtain a classical map on the complex plane

$$\Theta(\gamma)=\lim_{j\to\infty} \Theta_j(\gamma),$$

if the limit exists. Applying the stereographical projection we find a classical map $\tilde{\Theta}:S^2\to S^2$, which corresponds to the map $\Theta:C\to C$. The above method of assigning classical maps to a quantum dynamics built by SU(2) operators may be easily generalized\cite{92} to group SU($N$), with $N\geq 3$.

Let us examine the following simple example. Consider an operator describing the rotation along the $z$ axis by the angle $\kappa$:

$$U_1=\exp(i\kappa J_z)$$

Inasmuch as any CS$|j,\theta,\varphi\rangle$ is transformed into the rotated CS

$$U_1|j,\theta,\varphi\rangle=|j,\theta,\varphi+\kappa\rangle,$$

formula (47) does not depend on the quantum number $j$ and leads to the classical map given by
\[
\Theta(\gamma) = \Theta_j(\gamma) = e^{i\kappa \gamma}
\]  
(51)
or
\[
\tilde{\Theta}(\theta, \varphi) = (\theta, \varphi + \kappa).
\]  
(52)
Please note that there exist several other quantum operators corresponding to the same classical map in the sense of (47) and (48); one of the simplest might be written in the form
\[
U_2 = \exp(i(\kappa J_z + \lambda J_i / j)),
\]  
(53)
where $\lambda$ is a finite parameter and $J_i$ is an arbitrary component of the operator $J$.
Quantum map (49) transforms the coherent state $|j, \theta, \varphi\rangle$ into $|j, \tilde{\Theta}(\theta, \varphi)\rangle$. In order to verify validity of the condition (36) we calculate an integral over the circle of radius $\delta$ centered at $\tilde{\Theta}(\theta, \varphi)$:
\[
\int_{U(\Theta, \theta, \varphi, \delta)} |\langle j, \theta', \varphi' | U | j, \theta, \varphi \rangle|^2 \sin \theta' d\theta' d\varphi' = 1 - [\cos(\delta/2)]^{4j+2}.
\]  
(54)
The above result does not depend on the initial point of the phase space $(\theta, \varphi)$ and shows that in the semiclassical limit $j \to \infty$ the Husimi distribution of the quantum state $U_1 |j, \theta, \varphi\rangle$, integrated over a circle centered at the point $\tilde{\Theta}(\theta, \varphi)$ of an arbitrary small radius $\delta > 0$, tends uniformly to the unity. Requirement (36) is therefore fulfilled and the procedure of quantization linking the quantum map (49) with the classical map (52) is regular. Hence it is possible to apply to this case the correspondence principle for entropy (37). Classical map (52) describes an integrable system and is characterized by zero KS-entropy. It follows, therefore, that the quantum CS-entropy of the map (49) tends to zero in the limit $j \to \infty$.
Quantum map (49) can be easily generalized by admitting another unitary term parametrized by $\epsilon$:
\[
U_3 = \exp(i \kappa J_z) \exp(i \epsilon J_x / j).
\]  
(55)
This operator describes dynamics of the periodically kicked quantum top.\cite{93, 94, 95} For sufficiently large values of $\kappa$ and $\epsilon$ the classical analogue of this model displays chaos (the KS-entropy is positive\cite{96}) and thus might serve for further study on coherent states quantum entropy.
Setting the angle $\kappa$ to zero in the map (49) we obtain the identity operator $U = I$ and may calculate the entropy $H_{\text{mes}}$, which measures the randomness caused by a series of measurements performed by SU(2) coherent states instrument. Due to correspondence principle (37) we have
\[
\lim_{\hbar \to 0} H_{\text{mes}} = 0.
\]  
(56)
Preliminary results indicate\cite{97} that $H_{\text{mes}}$ tends to zero asymptotically as $-(\ln \hbar) \sqrt{\hbar}$.

VII. CONCLUDING REMARKS

We have proposed a general definition of the entropy of a dynamical system equipped with the measuring device. For a classical system and the sharp measuring instrument we obtain the Kolmogorov–Sinai entropy, which can be used to define classical chaos. Entropy of quantum system depends significantly on the way the process of a measurement (necessary to gain an information about the quantum dynamics) is performed.
Our approach covers older definitions of Pechukas–Beck–Graudenz\(^{20,30}\) (where the measuring device is the Lüders–von Neumann instrument) and of Helton–Tabor\(^{21}\) as the special cases. The other case corresponding to an approximate measurement of quantum observables (described by the coherent states instrument) is of a particular importance. Since the transition function used to describe the successive outcomes of this measurement has the well-defined semiclassical limit, the coherent states quantum entropy is related to the classical Kolmogorov–Sinai entropy. We have proved that the upper limit of the coherent states entropy as $\hbar \to 0$ is less than or equal to the Kolmogorov–Sinai entropy and we conjecture that the equality holds if the underlying classical system is sufficiently chaotic.

Two sources of randomness in our model (one coming from the measurement process and the other arising from the dynamics of the system) are separated by dividing the coherent states entropy into the measurement and the dynamical components. We believe that the dynamical coherent states entropy can be used to define and to measure quantum chaos. Note, however, that this quantity depends on the time between successive measurements and on the partition of the phase space. The character of this dependence requires a detailed study.

We hope that our work will be a starting point for further investigations in the subject. Hence we would like to point out some open problems that seem to be crucial for a better understanding of CS-quantum entropy.

1. Prove Conjectures 1 and 2 from the present paper. It may happen that the conjectures are false as stated and some additional assumptions are needed.
2. Analyze dependence of CS-quantum entropy (and dynamical CS-quantum entropy) on the time interval between two successive measurements (we expect that it is nonlinear) and on the partition of the phase space.
3. Study dependence of CS-quantum entropy on the choice of the set of coherent states. Note that we may choose different coherent states for a given physical system (e.g., vector coherent states, squeezed states).
4. Calculate the measurement CS-entropy for concrete examples, e.g., for SU(2) coherent states or torus coherent states.
5. Find an example of quantum map with positive dynamical CS-entropy (if Conjecture 2 is true, then any regularly quantized version of a hyperbolic and transitive classical map suits).
6. Examine whether the quantization procedures linking the frequently studied classical models (e.g., baker map, cat map, standard map, periodically kicked top) with their quantum counterparts fulfill the regularity condition (36). Compute analytically or numerically dynamical CS-entropy for these maps.
7. Check which results of the present paper may be generalized to Rényi CS-entropies.

Note added in proof. Pechukas–Beck–Graudenz entropy was studied earlier in the operational setting by Srinivas.\(^{103}\) His approach coincides with ours when the phase space is finite, but does not cover CS-entropy case. Lindblad\(^{104}\) (see also Appendix D) considered a Hilbert space version of Srinivas definition and proved an analog of our Proposition 3 in this situation. Mendes\(^{105}\) proposed a definition of quantum entropy that mimics the Brin–Katok\(^{106}\) approach to KS-entropy. However, he did not discuss the classical limit.

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APPENDIX A: PROOF OF PROPOSITION 1

Set \( p_t(a,b) = f(b)(T_t(\phi(a))) \) and \( q(u,b) = f(b)(T_{t_0}u) \) for \( t \in \Sigma, a,b \in \Omega, \) and \( u \in V \). We first prove the formula

\[
(\mathcal{T}_n(E_n) \circ \mathcal{T}_{n-1}(E_{n-1}) \circ \cdots \circ \mathcal{T}_0(E_0))u = \int_{E_n} \int_{E_{n-1}} \cdots \int_{E_0} p_{t_n-t_{n-1}}(y_{n-1},y_n) \cdots p_{t_1-t_0}(y_0,y_1)
\cdot q(u,y_0) \cdot T_{-t_n}(\phi(y_n))dm(y_0) \cdots dm(y_{n-1})dm(y_n),
\]

(A1)

by induction on \( n \in \mathbb{N} \).

Let \( n = 0 \). Applying Eqs. (11) and (14) we obtain

\[
(\mathcal{T}_0(E_0))u = T_{-t_0} \left( \int_{E_0} f(y_0)(T_{t_0}u) \phi(y_0)dm(y_0) \right) = \int_{E_0} q(u,y_0)T_{-t_0}(\phi(y_0))dm(y_0),
\]

(A2)

as required.

Assume the formula (A1) holds for \( n \); we shall prove it for \( n + 1 \). Applying again (11) and (14) we conclude that the left-hand side of the formula (A1) becomes

\[
T_{-t_n} \left( \int_{E_n} f(y_n) \left( \int_{E_{n-1}} \cdots \int_{E_0} p_{t_{n-1}-t_{n-2}}(y_{n-2},y_{n-1}) \cdots p_{t_1-t_0}(y_0,y_1)
\cdot q(u,y_0)T_{-t_{n-1}}(\phi(y_{n-1}))dm(y_0) \cdots dm(y_{n-1}) \phi(y_n)dm(y_n) \right) \right),
\]

(A3)

which is, clearly, equal to the right-hand side.

It follows from Eqs. (13) and (A2) that

\[
\mu_{t_0}^u(E_0) = \tau(\mathcal{T}_0(E_0))u = \int_{E_0} q(u,y_0)dm(y_0).
\]

(A4)

From (13), (A1), and (A4) we obtain

\[
\mu_{t_0 \cdots t_n}^u(E_0 \times \cdots \times E_n) = \int_{E_n} \int_{E_{n-1}} \cdots \int_{E_0} p_{t_n-t_{n-1}}(y_{n-1},y_n) \cdots p_{t_1-t_0}(y_0,y_1)q(u,y_0)
\times dm(y_0) \cdots dm(y_{n-1})dm(y_n)
\times dm(y_1) \cdots dm(y_{n-1})dm(y_n),
\]

(A5)

which is a desired conclusion.

APPENDIX B: PROOF OF PROPOSITION 2

Proof of Proposition 2(A): Let \( u \in V \) fulfill the condition

\[
T(\mathcal{T}(\Omega)u) = u.
\]

(B1)

Let \( E \in \Sigma \). Combining (11), (17), (20) and (B1) we obtain
\[ M^*(\mu_0^\alpha)(E) = \int_\Omega \left( \int_E P(a,b) \, dm(b) \right) d\mu_0^\alpha(a) \]
\[ = \int_E \left( \int_\Omega P(a,b) \, d\mu_0^\alpha(a) \right) dm(b) \]
\[ = \int_E \left( \int_\Omega f(b)(T(\phi(a))) : f(a)(u) \, dm(a) \right) dm(b) \]
\[ = \int_E f(b)(T(\mathcal{F}(\Omega)(u))) \, dm(b) \]
\[ = \int_E f(b)(u) \, dm(b) = \mu_0^\alpha(E), \quad \text{(B2)} \]

which was to be shown. Conversely, let \( u \) be a stationary state. Then
\[ \int_E f(b)(u) \, dm(b) = \int_E f(b)(T(\mathcal{F}(\Omega)(u))) \, dm(b) \quad \text{(B3)} \]
for every \( E \in \Sigma \). Hence
\[ f(b)(u) = f(b)(T(\mathcal{F}(\Omega)(u))) \quad \text{(B4)} \]
for almost every \( b \in \Omega \) (with respect to \( m \)). If \( \mathcal{F} \) is informationally complete, (B4) implies (B1), which completes the proof.

Proposition 2(B) follows immediately from (5), (22), and Proposition 2(A). Proposition 2(C) is obvious.

**APPENDIX C: GENERALIZED RÉNYI ENTROPY**

Let \( \mathcal{E}, \mathcal{A}, \) and \( u \) be as in Sec. III and let \( \alpha \neq 0 \). We define the Rényi entropy of order \( \alpha \) of \( \mathcal{E} \) in the initial state \( u \) with respect to the partition \( \mathcal{A} \) by the formula
\[ h^\alpha(\mathcal{E}, \mathcal{A}, u) := \begin{cases} h(\mathcal{E}, \mathcal{A}, u) & \text{for } \alpha = 1, \\ \limsup_{n \to \infty} \frac{1}{n} \ln(Z^\alpha) & \text{for } \alpha \neq 1, \end{cases} \quad \text{(C1)} \]
where the generalized partition function \( Z^\alpha \) is given by
\[ Z^\alpha := \sum_{i_0, \ldots, i_{n-1}} (\mu_0^{\alpha_i} \cdots \mu_0^{\alpha_{i_{n-1}}} (E_{i_0} \times \cdots \times E_{i_{n-1}}))^\alpha. \quad \text{(C2)} \]

Entropy of order 1 is by the definition the generalization of Shannon–Kolmogorov entropy, while entropy of order 0 is called topological entropy (we adopt the convention that \( 0^0 = 0 \)). It is easy to show that Rényi entropy is a nonincreasing function of \( \alpha \). Moreover, the number \( k \) of the elements of the partition \( \mathcal{A} \) gives the upper bound for the entropy. Namely,
\[ 0 \leq h^\alpha(\mathcal{E}, \mathcal{A}, u) \leq \ln k. \quad \text{(C3)} \]
In the analogy to (26) we can define measurement and dynamical Rényi entropy of order \( \alpha \).

**APPENDIX D: QUANTUM PROBABILITIES AND QUANTUM HISTORIES APPROACH**

Let \( P_i = P_i(t_0), \ i = 1, \ldots, k \), be arbitrary projection operators on Hilbert space \( \mathcal{H} (P_i^2 = P_i = P_i^\dagger) \) and let the transformed operator \( P_i(t_n) \) be equal to \( U^{t_n} P_i(t_0) U^{t_n} \), for any discrete time \( t_n \), where \( U \) is a unitary operator on \( \mathcal{H} \). For given initial state \( \rho \) any quantum history can be described by a sequence \( \{i_0, i_1, \ldots, i_{n-1}\} \), where \( i_j = 1, \ldots, k \) for \( j = 0, \ldots, n-1 \). The probability \( \mathcal{Q}_{i_0, \ldots, i_{n-1}} \) of this history is given by

\[
\mathcal{Q}_{i_0, \ldots, i_{n-1}} = \text{Tr}[P_{i_{n-1}}^\dagger(t_{n-1}) \cdots P_{i_1}^\dagger(t_1) P_{i_0}^\dagger(t_0) \rho P_{i_0}(t_0) P_{i_1}(t_1) \cdots P_{i_{n-1}}(t_{n-1})].
\]  

(D1)

Note a similarity between the above formula and the formula (13) for finite-dimensional distributions of quantum stochastic process applied to a family of transformed instruments (14). Formula (D1) might be considered as a special case of (13) obtained for the Lüders–von Neumann instrument defined by a family of orthogonal projection operators \( P_i \), and in fact, it has already appeared some 20 years before it was used by Griffiths et al. There is, however, an important difference between quantum histories approach and quantum measurement theory, as the former concerns mainly closed systems, contrary to the latter (for a comprehensive discussion, see the paper of Dowker and Halliwell).

In order to study semiclassical properties of a system and its behavior in the phase space Omnès and Halliwell proposed to use quasiprojectors, i.e., self-adjoint operators with discrete spectrum contained in the interval \([0, 1]\). Quasiprojectors can be defined by means of the coherent states \( |\alpha\rangle \)

\[
P_i^{CS} = \int_{E_i} |a\rangle \langle a| dm(a), \quad i = 1, \ldots, k,
\]  

(D2)

where \( \{E_1, \ldots, E_k\} \) is a partition of a phase space \( \Omega \). These operators describe the projection into the subsets \( E_i \) of the phase space, but do not satisfy the properties of projectors, since \( (P_i^{CS})^2 \neq P_i^{CS} \). inserting them directly to formula (D1), we obtain quantities denoted by \( \mathcal{Q}_{i_0, \ldots, i_{n-1}}^{CS} \).

Using these quantities and applying formula (31) Halliwell defined the notion of Shannon’s information of quantum history and derived lower bounds on Shannon’s information in particular cases. Note, however, that \( \mathcal{Q}_{i_0, \ldots, i_{n-1}}^{CS} \) cannot be considered as quantum probabilities, since, in contrast to (30),

\[
\sum_{i_0, \ldots, i_{n-1}} \mathcal{Q}_{i_0, \ldots, i_{n-1}}^{CS} \neq 1.
\]

(D3)

On the other hand, the choice

\[
\tilde{P}_i^{CS} = \left( \int_{E_i} |a\rangle \langle a| dm(a) \right)^{1/2}, \quad i = 1, \ldots, k,
\]

(D4)

may lead to a consistent interpretation, since these operators satisfy

\[
\tilde{P}_i^{CS} > 0, \quad i = 1, \ldots, k, \quad \text{and} \quad \sum_{i=1}^{k} (\tilde{P}_i^{CS})^2 = I.
\]

(D5)
which leads to the equality in formula (D3). For such operators Lindblad defined Shannon entropy using also formulas (D1) and (31) and denoted it by $H_n$. He remarked that the limit $h := \lim_{n \to \infty} H_n/n$ could be used as a measure of coherence in the system.

Omines also suggested considering as the quasiprojectors the pseudo-differential operators $P_1, \ldots, P_k$, defined by means of microlocal analysis. A similar idea has been used by Helton and Tabor, who defined

$$ Q_{i_0, \ldots, i_{n-1}} = \text{Tr}[\rho P_{i_0}UP_{i_1}UP \ldots UP_{i_{n-1}}]. \quad (D6) $$

These quantities need not be neither positive nor even real and can hardly be interpreted as probabilities.

Summing up, it seems that there is no simple way to describe successive approximate simultaneous measurements directly by formula (D1), and therefore to define the quantum entropy (31) we use formula (29) resulting from a more general formula (13).

**APPENDIX E: EQUIVALENCE OF (35) AND (36)**

The equivalence of conditions (35) and (36) follows from Ref. 43, Theorem 1.2. In the interest of self-contained exposition we present here a proof.

We first show that (36) implies (35). Let $g: X \to \mathbb{R}$ be a continuous function and $D_h = \langle \alpha_a|\alpha_a \rangle$ for every $a \in X$. Then

$$ \sup \left\{ \int_X g(b) \left| \frac{\langle \alpha_a^b | U_h | \alpha_a^b \rangle}{D_h} \right|^2 \text{dm}(b) - g(\Theta a) \right\} \leq \sup \left\{ \int_X f(b) - g(\Theta a) \right\} d\text{m}(b) $$

$$ \leq \sup \left\{ \sup_{a \in X} \sup_{b \in B(\Theta a, \delta)} |g(b) - g(\Theta a)| \right\} $$

$$ + 2\|g\| \sup \left\{ \int_X \frac{\left| \langle \alpha_a^b | U_h | \alpha_a^b \rangle \right|^2}{D_h} \text{dm}(b) \right\}. \quad (E1) $$

We can make the first term of the sum arbitrary small, as $g$ is uniformly continuous and $\delta$ is an arbitrary small real number. The second member tends to 0 as $h \to 0$. This gives (35).

We next prove that (35) implies (36). Suppose, contrary to our claim, that there exist $\epsilon > 0$ and a sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq X$, such that

$$ \int_{B(\Theta a_n, \delta)} \left| \frac{\langle \alpha_a^b | U_h | \alpha_a^b \rangle}{D_h} \right|^2 \text{dm}(b) < 1 - \epsilon \quad (E2) $$

for every $n \in \mathbb{N}$. Without loss of generality we can assume that $a_n$ tends to some $a \in X$ as $n \to \infty$. Take a continuous $g: X \to \mathbb{R}$ such that

$$ 0 \leq g \leq 1, $$

$$ g(x) = 0 \quad \text{for} \quad x \in B(\Theta a, 2\delta/3), $$

and

$$ g(x) = 1 \quad \text{for} \quad x \in B(\Theta a, \delta/3). \quad (E3) $$

Combining (E2) and (E3) we obtain, for sufficiently large $n \in \mathbb{N}$,
\[ \left| \int_X g(b) D_{\hbar}^{-1} | \langle \alpha_k^h | U_{\hbar} | \alpha_{a_n}^h \rangle |^2 \, dm(b) - g(\Theta a_n) \right| = 1 - \int_X g(b) D_{\hbar}^{-1} | \langle \alpha_k^h | U_{\hbar} | \alpha_{a_n}^h \rangle |^2 \, dm(b) \]

\[ \geq 1 - \int_{B(\Theta a, \delta/3)} D_{\hbar}^{-1} | \langle \alpha_k^h | U_{\hbar} | \alpha_{a_n}^h \rangle |^2 \, dm(b) \]

\[ \geq 1 - \int_{B(\Theta a, \delta)} D_{\hbar}^{-1} | \langle \alpha_k^h | U_{\hbar} | \alpha_{a_n}^h \rangle |^2 \, dm(b) \geq \varepsilon, \]

which contradicts (35).

**APPENDIX F: PROOF OF THE CORRESPONDENCE PRINCIPLE**

The correspondence principle can be easily derived from Refs. 43 and 74. We present here a direct proof based on Kifer's method.

Let \( n \in \mathbb{N} \). Let \( \mu_n \) and \( \mu_n^h \) be the \( n \)-dimensional distributions of the processes describing the evolution of the classical system and its quantum counterpart, respectively, i.e.,

\[ \mu_n(G_0 \times \cdots \times G_{n-1}) = m( \cap_{0 \leq j \leq n-1} \Theta^{-j}(G_j)), \]

and

\[ \mu_n^h(G_0 \times \cdots \times G_{n-1}) = \int \cdots \int \frac{|\langle a_i^h | U_{\hbar} | a_{i-1}^h \rangle |^2}{D_{\hbar}} \, dm(a_{n-1}) \cdots dm(a_0), \]

for \( G_0, \ldots, G_{n-1} \in B(X) \).

We demonstrate that

\[ \mu_n^h \to \mu_n \quad (\hbar \to 0), \]

where \( \to \) denotes the weak* convergence in the set of all probability measures on \((X, B(X))\).

Let \( g_0, \ldots, g_{n-1} \) be continuous maps from \( X \) to \( \mathbb{R} \). By (36) we obtain

\[ \int_{X^n} g_0(a_0) \cdots g_{n-1}(a_{n-1}) \, d\mu_n^h(a_0, \ldots, a_{n-1}) \]

\[ = \int_X \cdots \int_X \left( \prod_{i=1}^{n-1} \frac{|\langle a_i^h | U_{\hbar} | a_{i-1}^h \rangle |^2}{D_{\hbar}} g_i(a_i) \right) g_0(a_0) \, dm(a_0) \cdots dm(a_{n-1}) \]

\[ \hbar \to 0 \]

\[ \to \int_X g_0(a_0) g_1(\Theta a_0) \cdots g_{n-1}(\Theta^{n-1} a_0) \, dm(a_0) \]

\[ = \int_{X^n} g_0(a_0) \cdots g_{n-1}(a_{n-1}) \, d\mu_n(a_0, \ldots, a_{n-1}). \]

Applying the well-known theorem on weak* convergence (Ref. 102, Proposition 3.4.6) we deduce (F3).

Let \( 0 \leq i \leq k, j = 0, \ldots, n-1 \). Since \( m(\partial \mathcal{D}) = 0 \) and \( m \) is \( \Theta \)-invariant, we get
\[ \mu_n(\partial(E_{i_0} \times \cdots \times E_{i_{n-1}})) = 0. \]  
(\text{F5})

Using another theorem on weak* convergence (Ref. 102, Theorem 3.3.1) we conclude from (F3) and (F5) that

\[ \mu_n^\hbar(E_{i_0} \times \cdots \times E_{i_{n-1}}) \to \mu_n(E_{i_0} \times \cdots \times E_{i_{n-1}}) \quad (\hbar \to 0). \]  
(\text{F6})

It follows from Proposition 3 and (F6) that

\[ \lim_{\hbar \to 0} \sup_n H(U_{\hbar}, \mathcal{H}_\hbar, \mathcal{A}_\hbar, \rho_\hbar) = \lim_{\hbar \to 0} \inf_n \frac{1}{n} \sum_{i_0, \ldots, i_{n-1} = 1}^k \eta(\mu_n^\hbar(E_{i_0} \times \cdots \times E_{i_{n-1}})) \]

\[ \leq \inf \frac{1}{n} \sum_{i_0, \ldots, i_{n-1} = 1}^k \lim_{\hbar \to 0} \eta(\mu_n^\hbar(E_{i_0} \times \cdots \times E_{i_{n-1}})) \]

\[ = \inf \frac{1}{n} \sum_{i_0, \ldots, i_{n-1} = 1}^k \eta(\mu_n(E_{i_0} \times \cdots \times E_{i_{n-1}})) \]

\[ = \lim_{\hbar \to 0} \frac{1}{\hbar} \sum_{i_0, \ldots, i_{n-1} = 1}^k \eta(\mu_n(E_{i_0} \times \cdots \times E_{i_{n-1}})) \]

\[ = H(\Theta, \mathcal{A}) \leq H(\Theta), \]  
(\text{F7})

where \( \eta(x) = -x \ln x \) for \( x > 0 \) and \( \eta(0) = 0 \), which completes the proof.

52. J. T. Hoff, Generalized Coherent States and Their Applications (Springer-Verlag, Berlin, 1986).


P. Walters, An Introduction to Ergodic Theory (Springer-Verlag, New York, 1982).


M. Kuś (private communication, 1990).


W. Siomczyński and K. Życzkowski, Coherent states entropy of quantum measurements (to be published).


W. Siomczyński and K. Życzkowski, Coherent states entropy of quantum measurements (to be published).


