Universality of spectra for interacting quantum chaotic systems

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We analyze a model quantum dynamical system subjected to periodic interaction with an environment, which can describe quantum measurements. Under the condition of strong classical chaos and strong decoherence due to large coupling with the measurement device, the spectra of the evolution operator exhibit an universal behavior. A generic spectrum consists of a single eigenvalue equal to unity, which corresponds to the invariant state of the system, while all other eigenvalues are contained in a disk in the complex plane. Its radius depends on the number of the Kraus measurement operators and determines the speed with which an arbitrary initial state converges to the unique invariant state. These spectral properties are characteristic of an ensemble of random quantum maps, which in turn can be described by an ensemble of real random Ginibre matrices. This will be proven in the limit of large dimension.

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I. INTRODUCTION

Time evolution of an isolated quantum system can be described by unitary operators. Quantum dynamics corresponds then to an evolution in the space of quantum pure states since a given initial state $|\psi\rangle$ is mapped into another pure state $|\psi'\rangle=U|\psi\rangle$, where $U=\exp(-iH)$. Here $H$ represents a Hermitian Hamiltonian of the system and the time $t$ is set to unity.

If the underlying classical dynamics is chaotic the Hamiltonian $H$ or the evolution operator $U$ can be mimicked by ensembles of random unitary matrices [1,2]. In particular, spectral properties of an evolution operator of a deterministic quantum chaotic system coincide with predictions obtained for the Dyson ensembles of random unitary matrices [3]. The symmetry properties of the system determine which ensemble of matrices is applicable. For instance, if the physical system in question does not possess any time-reversal symmetry, one uses random unitary matrices of the circular unitary ensemble (CUE). If such a symmetry exists and the dimension of the Hilbert space is odd one uses symmetric unitary matrices of the circular orthogonal ensemble (COE) [4].

If the quantum system $S$ is not isolated, but it is coupled with an environment $E$, its time evolution is not unitary. One needs then to characterize the quantum state by a density operator $\rho$, which is Hermitian, $\rho=\rho^*$, positive, $\rho\geq 0$, and normalized, $\text{Tr} \rho=1$. Time evolution of such an open system can be described in terms of master equations [5], which imply that the dynamics takes place inside the set of quantum mixed states.

The coupling of the system $S$ with an environment $E$ can heuristically be described by adding to the Hamiltonian an anti-Hermitian component, $H\rightarrow H'=H-\imath\kappa WW^*$, where $W$ is an operator representing the interaction between both systems [6]. The corresponding ensembles of non-Hermitian random matrices with spectrum supported on the lower half of the complex plane were studied in [7,8]. For any positive value of the coupling strength parameter $\kappa$ the dynamics of the system is not unitary and eigenvalues of the evolution operator move from the unit circle inside the unit disk [see Fig. 1b]. A similar situation occurs if one takes into account dissipation in the system. Such a dynamics of eigenvalues of a nonunitary evolution operator in the complex plane was analyzed by Grobe et al. [9] and later reviewed by Haake [1].

Time evolution of an open quantum system can also be described in terms of a global unitary dynamics $V$, which couples together a system $S_A$ with another subsystem $S_B$, followed by averaging over the degrees of freedom describing the auxiliary subsystem. Technically, the image of an initial state $\rho$ of the system is obtained by a partial trace over the subsystem $S_B$, $\rho'=\text{Tr}_B[V(\rho\otimes \sigma)V^*]$, where $\sigma$ denotes the initial state of the environment. The map $\rho'=\Phi(\rho)$ defined in this way is completely positive and preserves the trace, so it is often called a quantum operation [10,11]. Note that in this approach both interacting subsystems $S_A$ and $S_B$ are set on an equal footing. The second system, usually referred to as an “environment,” is in fact treated symmetrically, and one may also consider a dual operation, in which the partial trace is taken over the principal subsystem $S_A$—compare Fig. 1c. A quantum map can be described by a superoperator $\Phi$, which acts on the space of density operators. If $N$ denotes the size of a density matrix $\rho$, the superoperator is represented by a matrix $\Phi$ of size $N^2$. In general such a matrix is not unitary, but obeys a quantum analogue of the Frobenius-Perron theorem, so its spectrum is confined to the unit disk [12]. Spectral properties of superoperators representing some exemplary interacting quantum systems were analyzed in [13–16]. It is worth to add that spectra of quantum superoperators are already experimentally accessible: Weinstein et al. [17] study spectra of superoperators corresponding to an NMR realization of exemplary quantum gates.

For a quantum operation $\Phi$ there exists an invariant state $\omega=\Phi(\omega)$. In a generic case of a typical (random) operation such an invariant state is unique [12]. If the action of the map is repeated $n$ times any initial quantum state $\rho$ converges to $\omega$ exponentially with the discrete time $n$. The rate of this convergence is governed by spectral properties of the superoperator, which can be characterized by the spectral gap,
panels\textsuperscript{chaos and strong decoherence these spectral properties are demonstrated that under the condition of strong classical evolution operators representing interacting quantum systems. We define as the difference between moduli of the two largest eigenvalues.

The main aim of this work is to analyze spectra of evolution operators representing interacting quantum systems. We demonstrate that under the condition of strong classical chaos and strong decoherence these spectral properties are universal and correspond to an ensemble of random operations\textsuperscript{[12]}. In other words, we explore the link between quantum chaotic dynamics and ensembles of random matrices. The analysis performed earlier for unitary quantum dynamics\textsuperscript{[1]} [see Fig. 1(a)] is extended for a more general case of nonunitary time evolution of interacting quantum systems. This problem can be described by an approach closely related to the one used earlier to characterize quantum dissipative dynamics. To describe spectra of such nonunitary evolution operators Grobe et al.\textsuperscript{[9]} applied random matrices of the complex Ginibre ensemble\textsuperscript{[18]}. The key idea of this work can be visualized in Fig. 2, which shows a bridge established between interacting quantum systems, appropriate ensembles of random operations and ensembles of Ginibre matrices. Since a superoperator describing one-step evolution operator can be represented as a real matrix\textsuperscript{[19]}, we are going to apply random matrices of the real Ginibre ensemble\textsuperscript{[20,21]}. In particular, we investigate time evolution of initially random pure states in a deterministic model of quantum baker map periodically subjected to quantum measurements, study the speed of their convergence to the invariant state and compare the results with those obtained for an appropriate ensemble of random operations. This paper is organized as follows. In Sec. II we introduce several versions of a deterministic model system: the quantum baker maps subjected to a measurement process. We analyze spectral properties of the corresponding evolution operator and investigate the spectral gap. A different, yet complementary approach is advocated in Sec. III, in which we study ensembles of random quantum operations. Our numerical results show that such ensembles can be useful to describe spectra of deterministic chaotic systems strongly interacting with an environment. In Sec. IV a relation between random quantum operations and ensembles of random matrices is analyzed. We prove that in the large $N$ limit the statistical properties of superoperators associated with random maps coincide with the predictions of the real Ginibre ensemble. The paper ends in Sec. V with concluding remarks which complete the reasoning sketched in Fig. 2. Necessary definitions of different classes of quantum maps are collected in the Appendix.

II. DETERMINISTIC SYSTEM: QUANTUM BAKER MAP SUBJECTED TO MEASUREMENTS

We start our work analyzing deterministic quantum systems which evolve periodically in time. In this section we introduce a generalized version of quantum baker map and investigate properties of the associated evolution operator. We shall concentrate on quantum dynamical systems, the classical analogues of which are known to be chaotic. Following the model of Balazs and Voros\textsuperscript{[22]} we consider the unitary operator describing the one-step evolution model of quantum baker map,

$$B = F_N^* \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix}.$$  \hspace{1cm} (1)

Here $F_N$ denotes the Fourier matrix of size $N$, $[F_N]_{jk} = \exp(\text{i}jk/2\pi N)/\sqrt{N}$, and it is assumed that the dimension $N$ of the Hilbert space $\mathcal{H}_N$ is even.

The standard quantum baker map $B$ may be generalized to represent an asymmetric classical map,
\[ B_K = F_{N}^{-1} \begin{bmatrix} F_{N/K} & 0 \\ 0 & F_{N-N/K} \end{bmatrix}, \]  

where \( K \equiv 2 \) is an integer asymmetry parameter chosen in such a way that the ratio \( N/K \) is integer. The standard model, obtained in the case \( K=2 \), corresponds to the classically chaotic dynamics characterized by the dynamical entropy \( H \) equal to \( \ln 2 \). This system can be considered as a two-dimensional lift of an one-dimensional nonsymmetric shift map,

\[ f_K(x) = \begin{cases} Kx/(K-1) : & x \in [0,(K-1)/K] \\ Kx - K + 1 : & x \in ((K-1)/K,1] \end{cases}. \]  

Chaos in such a system can be characterized by its dynamical entropy, equal to the mean Lyapunov exponent, averaged with respect to the invariant measure of the classical system. Since the uniform measure is invariant with respect to the map \( f_K \), the dynamical entropy \( h \) is equal to the mean logarithm of the slope \( df_K/dx \),

\[ h(K) = \frac{1}{K} \ln K + \frac{K-1}{K} \ln \frac{K}{K-1}. \]  

The entropy is maximal in the case \( K=2 \), while in the limit \( K \to \infty \) the entropy tends to zero. Hence the larger value of the parameter \( K \) is, the weaker chaos in the classical system becomes.

In the case of the quantum system acting on the \( N \)-dimensional Hilbert space the largest possible value of the asymmetry parameter reads \( K=N \). Thus the limiting case of the classically regular system cannot be obtained for any finite \( N \). The limit of vanishing dynamical entropy, \( h \to 0 \), can be approached only in the classical limit \( N \to \infty \) of the quantum system.

A generalized variant of a nonunitary baker map introduced by Saraceno and Vallejos described a dissipative quantum system [24]. In this work we study another model of noninvertible quantum baker map analyzed in [13,25], which is deterministic, conserves the probability, and is capable to describe projective measurements or a coupling with an external subsystem. Such a nonunitary dynamics can be represented as a quantum map and written in its Kraus form—for necessary definitions see the Appendix.

In general there exist \( M \) different outcomes of the measurement process and thus the map is described by a collection of \( M \) Kraus operators. The simplest nontrivial case of \( M=2 \) corresponds to dividing of the phase space into two parts, which we can choose to be the “lower” and the “upper” parts. Such a measurement scheme allows one to write down the quantum operation corresponding to the “sloppy baker map,” in which both pieces of the classical phase space are not placed precisely one by another, but in each step an overlap of a positive width takes place. In the classical model the upper piece of the phase space is shifted down by \( \Delta/2 \) [see Fig. 3(c)] so the invariant measure lives in the rectangle of the width \( (1-\Delta) \). To represent the shift in the quantum analogue of the map one uses a unitary translation operator \( V \) such that \( V^N = 1 \) and any momentum eigenstate \( |k \rangle \) is shifted by one, \( V|k \rangle = |k+1 \rangle \). Hence the shift down by \( \Delta/2 \) is realized by the unitary operator, \( V^{-\Delta/2} \). Thus the stochastic map describing the quantum sloppy map [13]

\[ \Phi_{B_K,\Delta}(\rho) = D_b B_K \rho B_K^\dagger D_b^\dagger + D_b B_K \rho B_K^\dagger D_b, \]  

consists of two Kraus operators, which act on the unitarily rotated state \( \rho' = B_K \rho B_K^\dagger \),

\[ D_b = F_{N/2} \begin{bmatrix} 1_{N/2} & 0 \\ 0 & 0 \end{bmatrix} F_N, \]  

\[ D_b = V^{-\Delta/2} F_{N/2} \begin{bmatrix} 0 & 1_{N/2} \\ 0 & 0 \end{bmatrix} F_N. \]  

The operator \( D_b \) describes the projection on the lower part of the phase space, while the definition of the operator \( D_b \) includes also the operator representing the shift of the upper domain down by \( \Delta/2 \). Observe that the parameter \( \Delta \) may take any real value from the unit interval [0,1]. However, the case \( \Delta=0 \) corresponds to the baker map without the shift but with a measurement, so it does not reduce to the standard unitary baker map \( B_K \).

One can also consider another classical model of double sloppy map, in which both domains are simultaneously shifted by \( \Delta/4 \) towards the center of the phase space [26] [Fig. 3(d)]. To write down the corresponding quantum model \( B_{K,\Delta} \) one needs thus to modify both Kraus operators, \( D_b \to V^{\Delta/4} D_b \) and \( D_b \to V^{-\Delta/4} D_b \).

Both variants of the model can be further generalized by allowing for a larger number \( M \) of measurement operators, represented by projectors on mutually orthogonal subspaces. For simplicity we assume here that the dimensionalities of all these subspaces are equal and read \( N/M \). Varying the parameter \( M \) one may thus control the degree of the interaction of the baker system with the environment and study the relation between the decoherence in the interacting quantum system and the spectrum of the corresponding superoperator.
The dimension of the Hilbert space $N=64$, parameter $M=2$, and the shift width $\Delta=1/4$ are kept fixed. A generic spectrum for $K=4$, $L=16$ is shown on panel (b). The subplots (a) and (c) are obtained for the cases of a weak classical chaos for $K=64$, $L=1$ and $K=L=32$, respectively, while the last case (d) shows the spectrum for the double sloppy map $B_{K,L,M}$ for $K=64$ and $L=64$.

Increasing the asymmetry parameter $K$ one can decrease the degree of classical chaos. To increase the degree of chaos one may just apply the quantum baker map twice, since the classical dynamical entropy of such a composite map is equal to $2 \ln 2$. In general one can allow for an arbitrary number of $L$ of unitary evolutions, and replace unitary $B$ by $B^2$. Alternatively one can say that the nonunitary measurement operation is performed only once every $L$ periods of the unitary evolution. Choosing the parameter $L$ to be of order of $N$ one can assure that the quantum dynamics is as “chaotic” as allowed by the quantum theory, what can be quantitatively characterized by the quantum dynamical entropy [27–29].

Therefore the generalized model of quantum sloppy baker map we are going to analyze here depends on the classical asymmetry parameter $K$, the width of the classical shift $\Delta$, the number of free evolutions $L$, and the quantum parameter $M$ denoting the number of measurement operators,

$$\Phi_{B_{K,L,M}}(\rho) = \sum_{m=1}^{M} D_m(B_K^L)^{L/2}(B_K^{L})^{L} D_m^\dagger.$$

Additionally, for each set of parameters of the model one can choose the appropriate set of projection operators $D_m$ which correspond to the shift applied on one or on two parts of the classical phase space.

Note that the measurement process can also be interpreted as an interaction with a measurement apparatus, described by an auxiliary Hilbert space of $M$ dimensions. Thus the model [Eq. (7)] represents an interacting quantum system and belongs to the general class of quantum operations defined by Eq. (A1). A rich structure of the model and the possibility to tune independently several parameters of the quantum system allows us to treat this model as a valuable playground to investigate spectral properties of superoperators, which represent nonunitary dynamics of interacting quantum systems.

We constructed quantum operations representing the generalized sloppy baker map [Eq. (7)] for several sets of the parameters of the model. In each case the superoperator $\Phi$ was obtained according to expression (A2) and diagonalized to yield the complex spectrum belonging to the unit disk.

In the case of several measurement operators, $M=2$, the quantum baker map represents a nonunitary dynamics. Under the condition of classical chaos the leading eigenvalue $z_1=1$ is not degenerated and all remaining eigenvalues are located inside the disk of the radius equal to the modulus of the subleading eigenvalue $R=|z_2|$.

The spectra of the superoperator of the generalized sloppy baker map [Eq. (7)] were found to depend weakly on the shift parameter $\Delta$. However, other parameters of the model (namely, $N$, $K$, $L$, and $M$) influence properties of the spectrum considerably (see Fig. 4).

As the asymmetry parameter $K$ increases the difference between the sizes of two domains which form the classical phase space becomes larger. In the extreme limit of
The classical system becomes only marginally chaotic, the eigenvalues are attracted to the unit circle and the spectral gap \( \gamma = 1 - |z| \) disappears.

On the other hand, if we increase the degree of the classical chaos by increasing the number \( L \) of unperturbed unitary evolutions, the size of the spectral gap does not change, but the spectrum fills the complex disc of radius \( R = |z| \) almost uniformly. Eventually, an increase in the number \( M \) of the measurements results in a faster decoherence in the system. This is reflected by an increase in the spectral gap \( \gamma \). In fact the radius \( R = 1 - \gamma \) of the disk supporting the spectrum decreases with \( M \) as \( M^{-1/2} \). This observation, demonstrated in Fig. 6, will be explained in Sec. IV.

III. ENSEMBLES OF RANDOM OPERATIONS

Let us pause for a while with the investigation of quantum deterministic systems and follow another approach. Not knowing much about a given chaotic system one can try to assume that the interactions are random and try to mimic statistical properties of the deterministic systems by appropriate ensembles of random matrices [1].

In this section we make a step in this direction and propose three different ensembles of random stochastic maps acting on the space \( M_{2N} \) of mixed states of size \( N \) with different physical interpretation. We assume that all unitary matrices \( U \) used below are drawn according to the Haar measure on the unitary group of corresponding dimension unless stated otherwise.

1. Environmental representation of a random stochastic map [12]. Choose a random unitary matrix \( U \) of composite dimension \( NM \) and construct a random map as

\[
\Phi_E(\rho) = \text{Tr}_E[U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger].
\]

It is assumed here that the environment, initially in an arbitrary pure state \( |\psi\rangle \) is coupled with the system \( \rho \) by a random global unitary evolution \( U \). The stochastic map is obtained by performing the partial trace over the \( M \)-dimensional environment.

2. Random external fields defined as a convex combination of \( M \) unitary evolutions [5]

\[
\Phi_R(\rho) = \sum_{m=1}^{M} \rho_m U_m \rho U_m^\dagger,
\]

where \( \rho_m \) are positive components of an arbitrary probabilistic vector of size \( M, \sum_{m=1}^{M} \rho_m = 1 \). All unitaries \( U_m \in U(N) \) are independent random Haar matrices. Random external fields form an example of bistochastic maps. They represent the physically relevant case in which the quantum system is subject to random unitary operations and can also be interpreted as quantum iterated function systems [30].

These maps, also called random unitary operations, were recently studied by Novotný et al., who investigated conditions under which the invariant state of the map is unique [31]. They found that the set of states attracted by a given invariant state depends on the algebraic properties of the unitaries \( U_m \) but is independent of the (nonzero) probabilities \( \rho_m \) present in Eq. (9). The probabilities determine the speed of convergence to the invariant state, and thus the size of the spectral gap but do not influence the structure of the attracting sets.

3. Projected unitary matrices acting on states of a composite dimension, \( N = KM \). All \( M \) Kraus operators are formed by unitarily rotated projection operators, \( A_m = P_m U \) for \( m = 1, \ldots, M \) which leads to the map

\[
\Phi_P(\rho) = \sum_{m=1}^{M} P_m \rho U \rho^\dagger P_m^\dagger.
\]

where \( U \) is a fixed random unitary matrix. Here \( P_m = P_m^\dagger = P_m^* \) denote projective operators on \( K \) dimensional mutually orthogonal subspaces, which satisfy the identity resolution, \( \oplus_m P_m = 1_N \). This ensemble of bistochastic maps corresponds to a model of deterministic quantum systems, in which unitary dynamics is followed by a projective measurement.

In the ensembles of random maps defined above the integer number \( M \) determines the number of Kraus operators and serves as the only parameter of each ensemble of random maps. Observe that in the special case \( M = 1 \) the dynamics reduces to the unitary evolution, so both variants of the model are used to describe quantum systems with or without a generalized antiunitary symmetry [1].

As shown in [12] the flat measure in the set of stochastic matrices is obtained for the coupling of the system with an environment of dimension \( M = N^2 \) so that the Choi matrix, \( D_\Phi = \langle \Phi \otimes I \rangle |\phi_i\rangle\langle\phi_i| \) of size \( N^2 \) has full rank. Here \( |\phi_i\rangle = \frac{1}{N} \sum_{j=1}^{N} |j,i\rangle \) represents the maximally entangled state on the bipartite Hilbert space, \( H_N \otimes H_N \). Due to the theorem of Choi the condition of complete positivity of the map is equivalent to positivity of the Choi matrix.

\[
\Phi \text{ is CP} \iff D_\Phi \succeq 0.
\]

In general the discrete parameter \( M \) characterizes the strength of the nonunitary interaction and we shall vary it from unity (unitary dynamics) to \( N^2 \), which describes a generic random stochastic map.

We have generated several realizations of random maps from the ensembles \( \Phi_E \) and \( \Phi_P \) introduced above. Exemplary spectra of superoperators for maps pertaining ensembles obtained for \( M = 2 \) are shown in Fig. 5. In the latter case we superimposed the spectra from two realizations of the map \( \Phi_P \) since by construction \( N^2/M \) eigenvalues of the superoperator are equal to zero.

In general, the spectra of random maps could be used to describe the spectra of deterministic system [Eq. (7)] under the condition of classical chaos and large decoherence. Numerical results performed for various models of quantum maps reveal an exponential decay of the mean trace distance [Eq. (A3)] to the invariant state. The average convergence rate is related with the size of the spectral gap \( \gamma \). As shown in Fig. 6 obtained for random operations as well as the generalized quantum baker map the radius \( R = 1 - \gamma \) of the disk in the complex plane, which contains all but the leading eigenvalue, decreases with the number of measurements as.
\[ R = |z_2| \sim \frac{1}{\sqrt{M}}. \] (12)

In the next section we present an explanation of this relation based on the theory of random matrices.

**IV. QUANTUM OPERATIONS AND REAL GINIBRE ENSEMBLE**

In this section we make a final link of our reasoning. Having showed a relation between deterministic chaotic systems which interact with an environment and random operations we make another step to describe spectral properties of superoperators associated with random operations by random matrices.

Looking at the spectrum of a random stochastic map \( \Phi_E \) shown in Fig. 5 one can divide the entire spectrum into three parts: (i) a single eigenvalue \( z_1 = 1 \), (ii) \( N^2 \) real eigenvalues distributed along the real line with a density \( P^\prime(x) \), and (iii) remaining \( N^2 \) complex eigenvalues \( z_i \), the distribution of which can be described by a density \( P^z(z) \) on a complex plane.

Any density operator \( \rho \) of size \( N \) can be represented using the generalized Bloch vector representation

\[ \rho = \sum_{i=0}^{N^2-1} a_i \lambda^i. \] (13)

Here \( \lambda^i \) denotes the generators of SU(\( N \)) such that \( Tr(\lambda^i \lambda^j) = \delta^{ij} \) and \( \lambda^0 \neq 1 \). Since \( \rho = \rho^\dagger \), \( a_i \in \mathbb{R} \) for \( i = 0, \ldots, N^2-1 \). The real vector \( [a_0, \ldots, a_{N^2-1}] \) is called the generalized Bloch vector. Thus

\[ \Phi(\rho) = \sum_i \left( \sum_j \Phi^j a_j \right) \lambda^i. \] (14)

The Bloch vector can also be used to represent an arbitrary operation. Using Kraus operators \( A^m \) one represents the element \( \Phi^j \) of the superoperator \( \Phi \) in a form

\[ \Phi^j = Tr[\lambda^j \Phi(\lambda^j)] = Tr \sum_m \lambda^m \lambda^j (A^m)^\dagger, \] (15)

where \( i, j = 1, \ldots, N^2-1 \). This square matrix of order \( N^2-1 \) will be called \( \mathbb{C} \). In a similar way we introduce the vector \( \kappa \) and find that the remaining elements of the matrix \( \Phi \) do vanish.

\[ \Phi^0 = Tr \sum_m \lambda^m \lambda^0 (A^m)^\dagger = Tr \lambda^0 \sum_m a_m (A^m)^\dagger = (\kappa), \] (16)

\[ \Phi^j = Tr \sum_m \lambda^m \lambda^j (A^m)^\dagger = Tr \lambda^0 \lambda^j = \delta^{ij}. \] (17)

Hence the superoperator \( \Phi \) can be represented as a real asymmetric matrix

\[ \Phi_{ij} = \begin{bmatrix} 1 & 0 \\ \kappa & \mathbb{C} \end{bmatrix}, \] (18)

where the \( N' = N^2-1 \) dimensional vector \( \kappa \) represents a translation vector while the \( N' \times N' \) real matrix \( \mathbb{C} \) is a real contraction [12]. Thus the spectrum of \( \Phi \) consists of the leading eigenvalue, equal to unity, and the spectrum of \( \mathbb{C} \).

Note that the complex eigenvalues of the real matrix \( \mathbb{C} \) appear in conjugated pairs, \( z \) and \( \bar{z} \), which is a consequence of the fact, that the map \( \Phi \) sends the set of Hermitian operators into itself, so as seen above the superoperator can be represented by a real matrix [19]. Since a map acting on states of size \( N \) is represented by a superoperator of dimension \( N^2 \) the following normalization relation holds, \( 1 + N^2 + N'' = N^2 \).

In the case that \( \mathbb{C} \) has only real eigenvalues one can bring \( \mathbb{C} \) by an orthogonal transformation \( O \) to lower triangular form

\[ \mathbb{C} = O(\Xi + \Lambda)O^{-1}, \] (19)

where \( \Lambda = \text{diag}(z_1, \ldots, z_{N'}) \) while \( \Xi \) has elements only below the diagonal. Thus

\[ d\mathbb{C} = O(O^{-1}dO(\Xi + \Lambda) - (\Xi + \Lambda)O^{-1}dO + d\Xi + d\Lambda)O^{-1}. \] (20)

Hence the measure \( d\mathbb{C} \) is given by

\[ D\mathbb{C} = \left| \prod_{i<j} (z_i - z_j) \prod_k d z_k \prod_{i<j} d\Xi_{ij} \right|, \] (21)

where the Vandermonde determinant is the Jacobian of the transformation from \( (O^{-1}dO)_{ij} \) to \( (O^{-1}dO)_{ij} \). Thus the measure \( d\mu(\Lambda) \) has the form

\[ D\mu(\Lambda) = \prod_{i<j} (z_i - z_j) \prod_k d z_k \Theta(D_{\Phi} \succeq 0), \] (22)

where in the last factor the positivity conditions of the corresponding Choi matrix is averaged by integration over the measure \( D\mathbb{C}d\Xi d(O^{-1}d\mathbb{C}) \). This factor is expected to be a smooth function of the eigenvalues \( z_1, \ldots, z_{N'} \). In the case that \( \mathbb{C} \) has a certain number of complex conjugate eigenvalues \( D\mu(\Lambda) \) is of similar form, but the product of differentials \( dz_k \) has to be interpreted as exterior product [21]. It turns out that for large dimension \( N' \) the measure \( d\mu(\Lambda) \) is given by the real Ginibre ensemble with the bulk of eigenvalues inside a certain disk in the complex plane. To prove this let us go back to the matrix representation of \( \Phi \) in terms of \( M \) Kraus operators \( A^m, m = 1, \ldots, M \). Then

\[ \Phi(\rho)_{ij} = \sum_{m=1}^{M} A^m_{il} \rho_{kl} (A^m)^* = \sum_{kl} \Phi_{ij,kl} \rho_{kl}, \] (23)

where \( A^m_{kl} \) are the matrix elements of Kraus operators and \( ^* \) denotes the complex conjugation; \( i, j, k, l = 1, \ldots, N \). The Kraus operators obey

\[ \sum_{m=1}^{M} \sum_{i=1}^{N} A^m_{il} (A^m)^* = \delta_{jl}, \] (24)

thus it is natural to assume that \( A^m_{kl} \) represent \( N \) columns of a matrix \( U \) drawn from a circular unitary ensemble of dimension \( NM \), i.e., \( U \in U(NM) \). Using formulas by Mello [32] for the first four moments of \( U(NM) \) we are able to find exactly the first two moments of matrix elements \( \Phi_{ij,kl} \). For example, for \( U \in U(N) \):

\[ \langle U_{bp} U_{aq}^* \rangle = \delta_{ab} \delta_{ap} / N. \] Here \( \langle \cdot \cdot \cdot \rangle \) means the av-

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average over the unitary group. This implies here
\[ \langle \Phi_{ij,kl} \rangle = \sum_{m=1}^{M} \langle A_{ik}^m (A_{jl}^m)^* \rangle = \frac{1}{N} \delta_{ij} \delta_{kl}. \] (25)

In this way we can derive the exact second moments:
\[ \langle \Phi_{ij,kl} \Phi_{ij,kl}^* \rangle = \frac{1}{(NM)^2 - 1} \left( M^2 \delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl} + M \delta_{ii} \delta_{kk} + M \delta_{ii} \delta_{kk} \delta_{ij} \delta_{kl} \right) \]
\[ - \frac{1}{NM} \left( \frac{1}{(NM)^2 - 1} \right) \left( M^2 \delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl} \right) \]
\[ + M \delta_{ii} \delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl}. \] (26)

We see that in the limit of large \( N \) the first two cumulants are identical to those of the Gaussian distribution with variance \( 1/(N^2 M) \)
\[ P(\Phi) \propto \exp \left( \frac{NM}{ik} \Phi_{ik,kl} - \frac{N^2 M}{2} \sum_{ijkl} |\Phi_{ijkl}|^2 \right). \] (27)

The factor 1/2 is due to the symmetry property \( \Phi_{ik,kl} = \Phi_{jk,li}^* \).
We will argue below that for large \( M \) in addition the higher cumulants can be neglected. Hence the superoperator \( \Phi \) associated with a random map can be described (up to the one eigenvalue 1) by the real Ginibre ensemble with eigenvalues inside a disk of radius \( 1/\sqrt{M} \), where \( M \) is the number of random Kraus operators defining \( \Phi \). This can also be seen by going back to the real matrix representation [Eqs. (15)–(18)].

Let us now argue that for large \( M \) we can neglect higher cumulants. First of all for large \( N \) the elements \( A_{ik}^m \), forming a minor of \( U \in \mathbb{U}(NM) \), are essentially independent Gaussian variables with zero mean and variance \( 1/NM \). Thus as a consequence of the central limit theorem for large \( M \) \( \Phi_{ik,kl} \) as sum of \( M \) essentially independent identically distributed variables is again Gaussian with variance \( M/(NM)^2 = 1/N^2 M \). Also in the bulk the different matrix elements of \( \Phi \) are independent. The average of \( \Phi_{ik,kl} \) is given by \( \delta_{ij} \delta_{kl}/N \).

To investigate the density of complex eigenvalues \( z = x + iy \) of the superoperator \( \Phi \), we analyzed their radial probability distribution \( P(r) \), where \( r = |z| \). The real eigenvalues are taken into account for this statistics. Figure 7 shows a comparison of numerical data obtained for several realizations of quantum baker map, projective random operations, and real random matrices pertaining to the Ginibre ensemble. The data are represented in the rescaled variable \( r_M = r/\sqrt{M} \) so that the radius of the disk of eigenvalues is set to unity. In all three cases displayed in the figure the radial density grows linearly, which corresponds to the flat distribution of eigenvalues inside the complex disk, in agreement with the predictions of the Ginibre ensemble. These results obtained for \( N=32 \) show a smooth transition of the density in the vicinity of the boundary of the disk at \( r_M = 1 \), which becomes more abrupt for larger \( N \). In the asymptotic case \( N \to \infty \), the density of rescaled eigenvalues is described by the circular law of Girko,
\[ P_r(z) \sim \Theta(1-|z|), \] (28)
derived for complex Ginibre matrices. The spectra of real random Ginibre matrices display a more subtle structure. A finite fraction of all eigenvalues are real, in analogy to the mean number of real roots of a real polynomial [33,34]. Real eigenvalues of a real Ginibre matrix cover the real axis with a constant density. Furthermore, for large dimension the density of complex eigenvalues is known to be asymptotically constant in the complex disk except for a small region near the real axis [20].

To check for what random operations these effects can be observed in the spectrum of the superoperator, we analyzed the average number \( \langle N^R \rangle \) of real eigenvalues of the superoperator \( \Phi \). For any realization of \( \Phi \) we have \( N^R = N_{\text{real}}/(N^2-1) \), where \( N_{\text{real}} \) is the number of real eigenvalues of the real matrix of size \( N' = N^2-1 \). These data are compared with predictions for the real Ginibre ensemble, hereafter denoted by \( \langle N^R \rangle_{RG} \). The following expression for the mean number of real eigenvalues of a real Ginibre matrix of size \( N^2-1 \) was derived in [35–37]
\[ \langle N^R \rangle_{RG} = 1 + \frac{\sqrt{2}}{\pi} \int_0^1 \frac{1}{t^{3/2}(1-t)} dt \]
\[ = \frac{\sqrt{2}}{\pi} \sqrt{N^2 - 1} \quad \text{as} \quad N \to \infty. \] (29)

These analytical results suggest to introduce a rescaled ratio
\[ \eta := \frac{\langle N^R \rangle_{\Phi}}{\sqrt{N^2 - 1}} \] (30)
to make easier a comparison of data obtained for various systems of size \( N \). Numerical results presented in Fig. 8 show that the superoperators associated with random maps are characterized by a nonzero fraction of real eigenvalues.

In the case of strong interaction with the ancilla of the size \( M = N^2 \) the dynamical matrix \( D_\Phi \) has full rank and the rescaled fraction of real eigenvalues of \( \Phi \) coincides with the prediction for the real Ginibre ensemble. To demonstrate fur-
Making use of the standard estimations rescaled formula \( R^N_1(\sqrt{2}) \) as a function of the matrix size \( N \). Solid horizontal line at \( \sqrt{2}/\pi \) represents the asymptotic value of the normalized ratio implied by Eq. (29).

ther spectral features characteristic of the real Ginibre ensemble, we analyzed spectra of superoperators and investigated the cross section of the probability distribution \( P(z) \) along the imaginary axis. Numerical data of this distribution denoted as \( P_1^R(y) \) obtained for an ensemble of random maps \( \Phi_E \) acting on the states of size \( N=3 \) are shown in Fig. 9.

In order to compare these data with predictions of the real Ginibre ensemble we need to assure a suitable normalization. Let us rescale the imaginary axis as \( y \rightarrow y_M = \sqrt{M} y \) so that rescaled formula (25) of [37] takes the form

\[
P_1^R(y_M) = R^N_1(\sqrt{M}) \simeq \frac{2M}{\pi} \exp(2M y_M^2) |y| \text{erfc}(|y| \sqrt{2M}).
\]

As shown in Fig. 9 these bounds are rather precise and describe well the numerically observed density \( P_1^R(y_M) \) of complex eigenvalues of the superoperators along the imaginary axis.

![Figure 8](image8.png)

**FIG. 8.** Rescaled ratio of the real eigenvalues \( \eta \) of the superoperator for random operations with \( M=N^2 \) (□), \( M=N \) (○), and real Ginibre matrices \( (\Delta) \) as a function of the matrix size \( N \). Solid horizontal line at \( \sqrt{2}/\pi \) represents the asymptotic value of the normalized ratio implied by Eq. (29).

![Figure 9](image9.png)

**FIG. 9.** Numerical data for the density \( P_1^C(y_M) \) of complex eigenvalues along the rescaled imaginary axis obtained for an ensemble of random operations \( \Phi_E \) of dimension \( N=8 \) and with parameter \( M=N^2=64 \) (●) are compared with lower and upper bounds [Eq. (33)] obtained for small \( |y| \) from the real Ginibre ensemble and represented by thick lines.

\[
\frac{1}{x + \sqrt{x^2 + 2}} < \exp(x^2) \int_{x}^{\infty} \exp(-t^2) dt \leq \frac{1}{x + \sqrt{x^2 + 4/\pi}}
\]

[see formula (7.1.13) at page 298 of Abramowitz and Stegun [38]], and the definition of the complementary error function \( \text{erfc}(z) \), one obtains from Eq. (32) an explicit form for lower and upper bounds for the rescaled distribution in the vicinity of the real axis

\[
\frac{1}{\pi} \frac{2}{1 + \sqrt{1 + \frac{1}{M y^2}}} \leq P_1^C(y_M) \leq \frac{1}{\pi} \frac{2}{1 + \sqrt{1 + \frac{2}{\pi M y^2}}}
\]

\[\text{(33)}\]

As shown in Fig. 9 these bounds are rather precise and describe well the numerically observed density \( P_1^C(y_M) \) of complex eigenvalues of the superoperators along the imaginary axis.

**V. CONCLUDING REMARKS**

In this work we analyzed spectra of nonunitary evolution operators describing exemplary quantum chaotic systems and the time evolution of initially pure states. We have chosen to work with a generalized model of quantum baker map subjected to measurements \([13,26]\), which allows one to control the degree of classical chaos and the strength of the interaction with the environment. The size of the quantum effects, proportional to the ratio of the Planck constant to the typical action in the system, is controlled by the size \( N \) of the Hilbert space used to describe the quantum system. The classical limit of the quantum model corresponds to the limit \( N \rightarrow \infty \).

Due to a quantum analogue of the Frobenius-Perron theorem the evolution operator has at least one eigenvalue equal to unity, while all other eigenvalues are contained in the unit disk in the complex plane. In a generic case the leading eigenvalue is not degenerated and the corresponding eigenstate represents the unique quantum state invariant with respect to the evolution operator.

Investigating the time evolution of initially random pure states we found out that in a generic case they converge exponentially fast to the invariant state. The rate of this relaxation to the equilibrium depends on the size of the spectral gap, equal to the difference between the moduli of the first and the second eigenvalues of the evolution operator. In particular, the relaxation rate \( \alpha \) was found to depend on the number of measurement operators \( M \) as \( \frac{1}{2} \ln M \).

Spectral properties of evolution operators of deterministic quantum systems interacting with the environment were compared with spectra of suitably defined ensembles of random matrices. Note that an idea to apply random matrices to model evolution operators of open deterministic quantum systems was put forward by Peplowski and Haake [39], but random maps used therein are not necessarily completely positive. This property is by construction fulfilled by the en-
A quantum stochastic map $\Phi$ sends the compact, convex set $\mathcal{M}_N$ of mixed density matrices into itself. Hence such a map has a fixed point, the invariant quantum state $\omega = \Phi(\omega)$. Thus the spectrum of any superoperator $\Phi$ representing a quantum operation contains an eigenvalue $z_1$ equal to unity, while all other eigenvalues belong to the unit disk. In the case of unitary dynamics the leading eigenvalue is degenerated, but for a random stochastic map the invariant state is generically unique, and the subleading eigenvalue satisfies $|z_2| < 1$. In this case any pure state, $|\psi\rangle\langle\psi|$, converges to the equilibrium state $\omega$ if transformed several times by the map $\Phi$. Note that this statement holds for a generic random operation, constructed according to the flat measure in the entire convex body of all quantum operations of a given size [12].

If the invariant state of a map $\Phi$ is unique one can characterize the convergence rate by the average trace distance of a random initial state to the invariant state $\omega$,

$$d(t) = \langle \text{Tr}[\Phi(\rho_0) - \omega]\rangle_{\Phi}. \quad (A3)$$

Here $t$ denotes the discrete time (i.e., the number of consecutive actions of a given map $\Phi$), while the average is taken over the ensemble of initially random pure states, $\rho_0 = |\psi\rangle\langle\psi|$. In the case of a generic quantum operation an exponential convergence to equilibrium, $d(t) = d(0)\exp(-\alpha t)$ was reported [12]. The convergence rate depends on spectral properties of the superoperator $\Phi$. The spectrum can be characterized by the spectral gap, $\gamma = 1 - |z_2|$, which generically determines the convergence rate, $\alpha = -\ln(1 - \gamma)$.

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APPENDIX: QUANTUM OPERATIONS AND SPECTRAL GAP

In this appendix we collect necessary definitions used in the main text. Define the set of quantum states $\mathcal{M}_N$ which contains all Hermitian positive operators $\rho$ of size $N \times N$ with trace set to unity. A quantum linear map $\Phi$ acting on $\mathcal{M}_N$ is called completely positive (CP) if positivity of the extended map, $(\Phi \otimes 1_M)(\rho) \succeq 0; \forall \rho$, holds for an arbitrary dimension $M$ of the extension and trace preserving (TP) if $\text{Tr}(\Phi(\rho)) = \text{Tr}(\rho)$. Any CP TP map is called quantum operation or stochastic map. If a quantum operation $\Phi$ preserves the identity, $\Phi(1/N) = 1/N$, the map is called bistochastic.

According to the dilation theorem of Stinespring [40] any CP map may be represented by a finite sum of $M$ Kraus operators,

$$\Phi(\rho) = \sum_{m=1}^{M} A^m \rho (A^m)^\dagger. \quad (A1)$$

If the Kraus operators $A^m$ satisfy the identity resolution, $\sum_m (A^m)^\dagger A^m = 1_N$, the map $\Phi$ is trace preserving. The corresponding superoperator can be expressed as a sum of the tensor products [10],

$$\Phi = \sum_{m=1}^{M} A^m \otimes (A^m)^\dagger, \quad (A2)$$

where the $^\dagger$ denotes the complex conjugation. Let $z_i$ with $i = 1, \ldots, N^2$ denote the spectrum of $\Phi$ ordered with respect to the moduli, $|z_1| \geq |z_2| \geq \cdots \geq |z_{N^2}| \geq 0$.

A quantum stochastic map $\Phi$ sends the compact, convex set $\mathcal{M}_N$ of mixed density matrices into itself. Hence such a map has a fixed point, the invariant quantum state $\omega = \Phi(\omega)$. Thus the spectrum of any superoperator $\Phi$ representing a quantum operation contains an eigenvalue $z_1$ equal to unity, while all other eigenvalues belong to the unit disk. In the case of unitary dynamics the leading eigenvalue is degenerated, but for a random stochastic map the invariant state is generically unique, and the subleading eigenvalue satisfies $|z_2| < 1$. In this case any pure state, $|\psi\rangle\langle\psi|$, converges to the equilibrium state $\omega$ if transformed several times by the map $\Phi$. Note that this statement holds for a generic random operation, constructed according to the flat measure in the entire convex body of all quantum operations of a given size [12].

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