Creep damage evolution equations expressed in terms of dissipated power

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Abstract

Evolution equations of continuum damage mechanics are usually expressed in terms of stresses; some authors introduce other quantities, like strains or elastic energy. The present paper introduces evolution equations expressed in terms of specific dissipated power: for uniaxial stationary creep such equations may simply be derived by some substitutions. It turns out that basing on available experimental data such an approach results in reduction of the number of parameters. Then the extensions to nonstationary creep and to multiaxial states are proposed as hypotheses subject to experimental verification. Finally, an extension to biomechanics is proposed, namely to biological materials subject to recovery. For such materials the damage rate is not necessarily positive: it may also be negative as a result of recovery. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The term Continuum Damage Mechanics (CDM) was proposed in 1977 by Janson and Hult [1], though the basic idea is due to Kachanov [2], 1958. Kachanov introduced a continuous scalar field variable \( \psi \) and called it “continuity”:

\[
\psi = \frac{A_{\text{net}}}{A_0},
\]  

(1)

where \( A_0 \) denotes the initial cross-sectional area and \( A_{\text{net}} \) – the real carrying area, decreased as a result of deterioration due to creep. Definition (1) holds for uniform state of stress; in the case of

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nonuniformity the areas $A$ should be replaced by elementary areas $\Delta A$. Further, Kachanov proposed for $\psi$ the following evolution equation valid in uniaxial tension characterised by the stress $\sigma$: 

$$\dot{\psi} = - C \left( \frac{\sigma}{\psi} \right)^v,$$  

(2) 

where $C$ and $v$ are material constants essentially depending on temperature, and dot denotes differentiation with respect to time $t$. The variable $\psi$ starts from $\psi = 1$ (unless some other factors result in a certain value $\psi_0$ this problem will be discussed in Section 6) and decreases to $\psi = 0$; the last value denotes total rupture or initiation of a rupture front (macro-damage).

Odqvist and Hult [3] introduced a more convenient variable

$$D = 1 - \psi = \frac{A_0 - A_{net}}{A_0},$$  

(3)
and called it “damage”; it starts from $D = 0$ (or a certain value $D_0$) ending with $D = 1$. The evolution Eq. (2) was generalised to

$$\dot{D} = f(\tilde{\sigma})$$

where $f$ is a non-decreasing function, and

$$\tilde{\sigma} = \frac{\sigma}{1 - D} = \frac{P}{A_{net}}$$

was called by them the “real stress”. Now the term “effective stress” for $\tilde{\sigma}$ is in common use ([4–6], generalisation to a tensorial damage parameter), though this term may be misleading since it is also being used for reduced stress in multiaxial cases. The symbol $P$ in Eq. (5) denotes tensile force. Odqvist and Hult [3] also showed that if the power function (2) is assumed, then the results of Kachanov’s approach coincide with an earlier theory of cumulative creep damage proposed by Robinson [7].

The theoretical maximal value of $D$, namely $D = 1$, is often being replaced by a certain smaller value $D_{cr}$ in view of at least two reasons. First, $D_{cr}$ may be evaluated in connection with Griffith’s critical length of a crack; second, numerical solution of particular problems up to $D = 1$ may be difficult because of singularities of evolution equations in this case. Considering the first limitation Janson and Hult [1] found for elastic-brittle materials $D_{cr} = 0.5$, whereas for ductile-plastic materials Lemaitre [8] proposed $D_{cr} = 0.85$. In view of numerical difficulties Murakami et al. [9] proposed $D_{cr} = 0.99$. They also showed that the effects of the value of $D_{cr}$ on rupture times are rather small; in this paper we assume $D_{cr} = 1$.

Rabotnov [4] assumed that damage affects also the strain rates $\dot{\varepsilon}$ and proposed the following constitutive equations of creep and evolution equations for damage:

$$\dot{\varepsilon} = C_1 \sigma^n (1 - D)^{-m},$$

$$\dot{D} = C_2 \sigma^\nu (1 - D)^{-\mu}$$

with six material constants $C_1$, $C_2$, $n$, $m$, $\nu$, $\mu$, depending on the temperature. Strictly speaking stationary (secondary) creep takes place just in the case $m = 0$; if $m \neq 0$ then the tertiary creep is described, or rather partly described since changes of the total cross section due to ductile effects are not included. We shall call the case $m \neq 0$ the “quasi-stationary creep”; these effects are important, since usually creep rupture is preceded by tertiary creep. In principle, all four exponents are regarded as independent subject to experimental evaluation. However, if we assume the concept of the real-effective stress $\tilde{\sigma}$, Eq. (5) and suppose that in all physical relations $\sigma$ should be simply replaced by $\tilde{\sigma}$, then $m = n$, $\mu = \nu$, and just two exponents remain as free. Moreover, Hayhurst [10] analysed the experimental data gathered by Odqvist [11] for stationary creep in various materials and temperatures and noticed that in most cases the equation

$$\nu = 0.7n$$

is sufficiently accurate.
A considerable number of papers is devoted to generalisations of the evolution equation (6) for nonstationary creep, multiaxial stress, viscoplasticity, interaction of brittle and ductile creep rupture, interaction of damage due to creep and fatigue, several scalar damage variables, tensorial damage variables as well as to their geometrical and physical interpretation etc.; we mention here just the surveys written by Hult [12], Krajcinovic [13], Murakami [14], Chaboche [15], and the books by Kachanov [16], Lemaitre [17], Krajcinovic [18] and Skrzypek and Ganczarski [19]. Papers on structural optimization allowing for CDM were reviewed by Życzkowski [20]. Many optimal solutions were quoted by Skrzypek and Ganczarski [19].

The aim of the present paper is to express evolution equations of CDM in terms of specific (unit) dissipated power. The basic formula for stationary creep in uniaxial tension will be derived and then their extension to more general cases will be postulated as a hypothesis subject to experimental veriﬁcation. Such an approach will result in reduction of the number of material constants, and this is always important in applications. Finally, an extension to biomechanics will be given, namely a damage evolution equation for biological materials allowing for recovery will be proposed.

2. Various forms of CDM evolution equations

Basic evolution equations for the damage parameter \( D \) (4), (6), present the derivative \( \dot{D} \) as a function of the stress \( \sigma \) in uniaxial case or of certain stress invariants in multiaxial case. This approach is natural since most processes are stress controlled. However, sometimes introduction of other variables into evolution equations is necessary and sometimes it is simply convenient to reduce the number of free parameters, as it will be shown in the present paper.

Several papers express the derivative \( \dot{D} \) in terms of strains. The first proposal is due to Rabotnov [4]; he eliminated stresses from the constitutive equation of stationary creep in exponential form and derived the formula

\[
\dot{D} = C \exp(k\varepsilon)(1 - D)^{-\mu}.
\]  

(8)

He also introduced the derivative \( dD/d\varepsilon \) instead of \( dD/dt \). Belloni et al. [21] noticed that the damage variable \( D \) may also be expressed in terms of density variation;

\[
D = -\frac{\Delta \rho}{\rho_0},
\]

(9)

where \( \rho_0 \) is the initial density of the material and \( \Delta \rho \) the (negative) increase of this density due to damage. They also proposed the evolution equation in the finite form (for constant stress)

\[
D = He^a \exp\left(-\frac{\beta}{T}\sigma^a t^b\right),
\]

(10)

performed several series of experiments, and noticed that in this equation the variable \( \varepsilon \) is the most important, so in the first approximation the equation

\[
D = He^a
\]

(11)
may be proposed. This idea was extended to variable stress by Cozzarelli and Bernasconi [22] and Lee et al. [23], who proposed differential evolution equation in the form

\[
\dot{D} = f(\dot{e}, e, D, \sigma).
\]  

(12)

Hayhurst et al. [24] noticed that if a function of time appears in the evolution equation in the form of a multiplier \( f(t) \), it may be eliminated by a suitable transformation.

Litewka [25] expressed his damage evolution equation in terms of specific elastic strain energy for a homogenised equivalent material possessing the same elastic properties as the damaged solid \( \Phi_e \). Considering tensorial damage parameter \( \Omega \) he proposed the equation

\[
\dot{\Omega} = B_1 \Phi_e^{m} I + B_2 \Phi_e^{n} \sigma^*,
\]  

(13)

where \( I \) denotes the unit tensor, \( \sigma^* \) - the modified stress tensor, \( m, n, B_1 \) and \( B_2 \) are the material constants. In practical applications he assumed \( B_1 = 0 \) and \( n = 2 \); this value corresponds to \( v = 5 \) in Kachanov’s approach.

A general approach to constitutive equations and damage evolution equations based on thermodynamical considerations is due to Krajcinovic [26], Lemaitre and Chaboche [27]. If we express the Helmholtz free energy of damaged material \( \psi \) in terms of elastic strains \( e_i \), hardening parameters \( \alpha_i \) and damage parameters \( D_i \), then the thermodynamic forces \( R_i \) conjugate with \( D_i \) are given by

\[
R_i = \rho_0 \frac{\partial \psi}{\partial D_i}.
\]  

(14)

Further, postulating the existence of the flow potential \( F \) we obtain the evolution equations for damage parameters \( D_i \) in the form

\[
\dot{D}_i = \frac{\partial F}{\partial R_i}.
\]  

(15)

The authors give some examples of application of this approach and comparison with Kachanov’s formula.

3. CDM evolution equations for uniaxial stationary creep in terms of dissipated power

The first idea to express creep rupture in terms of specific dissipated power is due to Olszak and Bychawski [28,29]. They proposed the creep rupture criterion in the form

\[
F(\Phi_e, \Psi) = \text{const.},
\]  

(16)

where \( \Psi \) denotes specific dissipated power,

\[
\Psi = \sigma_{ij} \dot{e}_{ij}.
\]  

(17)
\( \varepsilon_{ij} \) denote just inelastic strains, and summation convention holds. The authors used the notation \( W_B \), but this symbol like various other symbols used in the literature (\( \mathcal{A}, \mathcal{U}, L, P, D, \Phi, Y \)) seem inconvenient, so we introduced here the notation \( \Psi \). Phenomenological equation (16) reflects rather the primary idea of the authors, but in the applications they used dissipated energy instead of the power and this step finds easy justification. Their equation was not connected with CDM; an attempt to combine it with damage mechanics is due to Walczak [30].

However, it turns out that the introduction of specific dissipated power into CDM evolution equations proves useful since it may result in reduction of number of material constants and enables interesting generalisations. In the case of uniaxial stationary creep and most often used power functions (6) we have

\[
\Psi = C_1 \sigma^{n+1}(1 - D)^{-m},
\]

hence

\[
\sigma = [C_1^{-1} \Psi (1 - D)^m]^{1/(n + 1)}
\]

and after substitution into the second Eq. (6)

\[
\dot{D} = C_2 (C_1^{-1} \Psi)^{(v/(n + 1))}(1 - D)^{(m v/(n + 1)) - \mu}.
\]

Odqvist [11] gives a table of experimentally evaluated constants \( h \) and \( v \) for four steels at various temperatures, 13 data in total (his table contains data for 15 metals, but just for those four steels the exponent \( v \) is given). These data are gathered in Table 1 with added values of the ratio \( v/(n + 1) \). It turns out that the ratio is close to 0.5 and the mean value for 13 cases amounts to 0.541. So, in what follows we assume that the following relation between \( n \) and \( v \) holds for a relatively broad class of materials and temperatures in uniaxial creep:

\[
v = \frac{n + 1}{2}.
\]

From among more recent experimental results we quote the pair \( n = 6.95, v = 5.52 \) given by Trąmpezyński et al. [31] for tough pitch high conductivity copper B.S. 2873-CID at 250°C, two pairs obtained by Hayhurst et al. [24]: \( n = 6.90, v = 6.48 \) for D19S aluminium alloy at 150°C, \( n = 1.737, v = 0.478 \) for 316 stainless steel at 550°C and the pair \( n = 2.97, v = 1.21 \) for extruded copper bar at 250°C obtained by Hayhurst et al. [32]. Essential differences in the exponents for copper at 250°C in Refs. [31, 32] may partly be explained by various assumptions: in Ref. [31] the authors assumed \( \mu = v \) whereas in Ref. [32] \( \mu \neq v \) namely \( \mu = 3.83 \), and additionally a time factor \( t^{-0.79} \) was allowed for. In any case these differences clearly visualise difficulties of experimental evaluation of creep constants. Accuracy of Eq. (21) for the data given above is much worse than in the case of Odqvist’s table; nevertheless, the mean value of the ratio \( v/(n + 1) \) for these four pairs amounts 0.498, and though high accuracy of this result (in comparison to 0.5) is doubtlessly obtained by accident, Eq. (21) may serve at least as a useful approximation. Considerations of the present paper will be based on Eq. (21).

Introducing Eq. (21) into Eq. (20) we rewrite this evolution equation in much simpler form:

\[
\dot{D} = \frac{\sqrt{\Psi}}{C_d (1 - D)^{(\mu - m)/2}}.
\]
Table 1
Material constants of creep at various temperatures (after Odqvist [11])

<table>
<thead>
<tr>
<th>Material</th>
<th>Temperature (°C)</th>
<th>n</th>
<th>(v)</th>
<th>(\frac{v}{(n+1)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Carbon steel (rolled) 450</td>
<td>5</td>
<td>3.5</td>
<td>0.583</td>
<td></td>
</tr>
<tr>
<td>0.40 Mn, 0.25 Si, 0.12 C 500</td>
<td>3.3</td>
<td>2.3</td>
<td>0.534</td>
<td></td>
</tr>
<tr>
<td>550</td>
<td>2.5</td>
<td>1</td>
<td>0.286</td>
<td></td>
</tr>
<tr>
<td>2. Low alloy steel (rolled) 550</td>
<td>4.15</td>
<td>3</td>
<td>0.583</td>
<td></td>
</tr>
<tr>
<td>1.0 Cr, 0.75 Mn, 0.35 Mo, 0.25 Si, 0.15 C 600</td>
<td>2.74</td>
<td>2</td>
<td>0.535</td>
<td></td>
</tr>
<tr>
<td>650</td>
<td>2.1</td>
<td>1.5</td>
<td>0.484</td>
<td></td>
</tr>
<tr>
<td>3. Stainless-steel (rolled) 450</td>
<td>6.5</td>
<td>4.5</td>
<td>0.600</td>
<td></td>
</tr>
<tr>
<td>18 Cr, 8 Ni, 0.45 Si, 0.4 Mn, 0.1 C 500</td>
<td>5.6</td>
<td>3.8</td>
<td>0.591</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>4.5</td>
<td>3.1</td>
<td>0.564</td>
<td></td>
</tr>
<tr>
<td>650</td>
<td>4.0</td>
<td>2.8</td>
<td>0.560</td>
<td></td>
</tr>
<tr>
<td>700</td>
<td>3.5</td>
<td>2.5</td>
<td>0.555</td>
<td></td>
</tr>
<tr>
<td>4. Stainless-steel (rolled) 550</td>
<td>6</td>
<td>4</td>
<td>0.571</td>
<td></td>
</tr>
<tr>
<td>25 Cr, 20 Ni, 1.5 Si, 1 Mn, 0.1 C 600</td>
<td>5</td>
<td>3.5</td>
<td>0.583</td>
<td></td>
</tr>
<tr>
<td>650</td>
<td>5</td>
<td>3.5</td>
<td>0.583</td>
<td></td>
</tr>
<tr>
<td>700</td>
<td>5</td>
<td>3.5</td>
<td>0.583</td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>5</td>
<td>3.5</td>
<td>0.583</td>
<td></td>
</tr>
</tbody>
</table>

Mean value of \(\frac{v}{(n+1)}\)

\[0.541\]

where \(C_d = \sqrt{C_1/C_2}\); the constant \(C_d\) (damage modulus) is located here in the determinator, since then the increase of modulus corresponds to increasing strength of the material. The dimension of \(C_d\) is equal to the square root of the dimension of the product stress × time.

We now consider two particular cases of Eq. (22). If we assume stationary, secondary creep, then \(m = 0\) and

\[
\dot{D} = \frac{\sqrt{\Psi}}{C_d(1 - D)^{n}}.
\] (23)

On the other hand, if we assume the concept of real-effective stress and quasi-stationary tertiary creep, then \(\mu = v = (n + 1)/2\), \(m = n\), and we obtain

\[
\dot{D} = \frac{1}{C_d} \sqrt{\frac{\Psi}{1 - D}}.
\] (24)
The evolution equation (24) may be recommended because of two reasons: first, creep rupture is practically always preceded by tertiary creep, and second, Eq. (24) is particularly simple, just with one material constant, namely modulus $C_d$ depending on the temperature. The square root of $\Psi$ in the formulae (22)–(24) is not particularly surprising, since similar root appears, for example, in all the formulae for reduced stress based on energy failure hypotheses (Beltrami, Huber–Mises–Hencky), $\sigma_{red} = c\sqrt{\Phi}$. On the other hand there is no direct analogy, since deriving the formulae for reduced stress we base on linear elasticity, whereas here we consider highly nonlinear materials.

It should also be noted that for damaged materials Sidoroff [5] introduced the concept of “damaged complementary energy”, replacing in classical formula the stress $\sigma$ by the real-effective stress $\hat{\sigma}$, (5). Going this way, we can also introduce the “damaged specific dissipated power” $\Psi_d$ by the formula

$$\Psi_d = \hat{\sigma}_{ij} \hat{\varepsilon}_{ij} = \frac{\sigma_{ij} \varepsilon_{ij}}{1 - D}$$

and rewrite Eq. (24) in an even simpler form

$$\dot{D} = \frac{1}{C_d} \sqrt{\Psi_d}$$

simplicity of this equation is rather apparent, since the damage parameter $D$ is also hidden in $\Psi_d$.

According to the definition the function $\Psi$ in Eq. (23) does not depend on $D$, so we can separate the variables and integrate. Imposing the initial condition $D = 0$ for $t = 0$ (other forms of this condition will be discussed in Section 6) we obtain

$$\frac{1}{\mu + 1} [1 - (1 - D)^{\mu + 1}]C_d = \int_0^{t_r} \sqrt{\Psi(t)} \, dt,$$

where $t$ is the variable of integration. Hence the general formula for $D$ looks as follows:

$$D = 1 - \left[ 1 - \frac{\mu + 1}{C_d} \int_0^{t_r} \sqrt{\Psi(t)} \, dt \right]^{1/(\mu + 1)},$$

and assuming $D = 1$ we obtain for the creep rupture time $t_r$ the condition

$$\int_0^{t_r} \sqrt{\Psi(t)} \, dt = \frac{1}{\mu + 1} C_d.$$

In the case of nonuniform state of stress Eq. (29) determines the time of onset of macro-crack propagation. This formula resembles that proposed by Olszak and Bychawski [28,29] but the difference is essential: calculating dissipated energy we integrate $\Psi(t)$, whereas in Eq. (29) $\sqrt{\Psi(t)}$ is integrated.

If we make an additional assumption (not conforming to the derivation given above for quasi-stationary tertiary creep) that the dissipated power $\Psi$ in Eq. (24) does not depend on $D$, then we may use the solutions (27)–(29) with purely formal substitution $\mu = 1/2$. 
4. Extension to non-stationary uniaxial creep and viscoplasticity

For stationary uniaxial creep and damage described by power laws (6) the evolution equation (24) was simply derived under additional condition (21) which was assumed to hold for various materials and temperatures. Now, we may formulate a hypothesis that Eq. (24) is also valid for some non-stationary creep processes and for multiaxial states. Such hypotheses are, of course, subject to experimental verification. A comparison with existing experimental data for non-stationary creep is rather difficult in view of a great variety of forms of constitutive and evolution equations used by various authors.

First, we consider combined strain-hardening creep (Davenport–Rabotnov) with time-hardening creep

\[ \dot{\varepsilon} = C_1 \sigma^n \varepsilon^{-\varepsilon t^\delta}. \]  

(30)

In this case

\[ \Psi = C_1 \sigma^{n+1} \varepsilon^{-\varepsilon t^\delta} \]  

(31)

and substituting into Eq. (23) we obtain

\[ \dot{D} = \frac{\sqrt{C_1}}{C_d} \sigma^{(n+1)/2} \varepsilon^{-(x/2)t^{\delta/2}}(1 - D)^{-\mu}. \]  

(32)

On the other hand, to be compatible with the derivation of Eq. (24) we have to replace in Eq. (30) \( \sigma \) by \( \tilde{\sigma} \), Eq. (5),

\[ \dot{\varepsilon} = C_1 \sigma^n \varepsilon^{-\varepsilon t^\delta(1 - D)^{-n}}, \]  

(33)

and now Eq. (24) yields

\[ \dot{D} = \frac{\sqrt{C_1}}{C_d} \sigma^{n+1/2} \varepsilon^{-x/2t^{\delta/2}}(1 - D)^{-(n+1)/2}. \]  

(34)

If we suppose \( \mu = \nu = (n + 1)/2 \), then Eqs. (32) and (34) are equivalent, but in effective calculations the differences appear, since \( \varepsilon \) in these equations is determined by different formulae, namely, in Eq. (34) it depends also on \( D \).

Using Eq. (32) we can separate the variables and integrate

\[ \frac{1}{\mu + 1} [1 - (1 - D)^{\mu + 1}] = \frac{\sqrt{C_1}}{C_d} \int_0^\tau \sqrt{\sigma^{n+1}(\tilde{t}) \varepsilon^{-x(\tilde{t}) \tilde{t}^\delta}} \, d\tilde{t}. \]  

(35)

The creep rupture condition looks as follows:

\[ \int_0^\tau \sqrt{\sigma^{n+1}(t) \varepsilon^{-x(t)} t^\delta} \, dt = \frac{C_d}{(\mu + 1)\sqrt{C_1}}. \]  

(36)
If we now confine our considerations to a constant-stress test, equivalent here to a constant-load test (if ductile effects of decreasing cross-sectional area are neglected), then in Eq. (30) the variables can be separated and integrated effectively:

$$\varepsilon = \left[ \frac{z + 1}{\delta + 1} C_1 \sigma^n t^{\delta + 1} \right]^{1/(\delta + 1)}.$$  

(37)

and condition (36) takes the form

$$t_r = \frac{\bar{C}}{\sigma} \left( \frac{n + \varepsilon + 1}{\delta + \varepsilon + 2} \right).$$  

(38)

where all the constants are hidden in $\bar{C}$. For $\varepsilon = \delta = 0$ (stationary creep) we obtain

$$t_r = \frac{\bar{C}}{\sigma} \left( \frac{n + 1}{2} \right).$$  

(39)

and in view of Eq. (21) we arrive at the classical Kachanov’s formula.

Use of Eq. (34) is more difficult here, since Eqs. (33) and (34) are coupled via both $\varepsilon$ and $D$ and the uncoupling problem is more complicated. Detailed calculations will not be given here.

As an example of application to viscoplasticity we make use of the Perzyna [33] equation

$$\dot{\varepsilon} = C\left( \frac{\sigma}{\sigma_0} - 1 \right)^n,$$  

(40)

where the angular brackets denote

$$\langle x \rangle = \begin{cases}  
x & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0, 
\end{cases}$$  

(41)

and $\sigma_0$ is the yield-point stress. Under the assumption $\sigma > \sigma_0$ we have

$$\Psi = \frac{C}{\sigma} \left( \frac{\sigma}{\sigma_0} - 1 \right)^n.$$  

(42)

Introduction of the real-effective stress (5) brings here some difficulties, so we left $\sigma$ without correction and use Eq. (23) with $m = 0$

$$\dot{D} = \frac{\sqrt{\sigma (\sigma - \sigma_0)^n}}{\bar{C}(1 - D)^m}.$$  

(43)

This equation resembles that proposed by Lemaitre and Chaboche [34] where also a threshold value of $\sigma$ is introduced.

## 5. Extension to multiaxial states

Specific dissipated power Eq. (17) is an invariant and hence an extension of the evolution Eqs. (23) and (24) to multiaxial states is direct: we can simply postulate these equations to be valid in multiaxial states as well. Simplicity of those equations remains, but generality is very restricted in this case.
Let us use the Norton–Odqvist constitutive equations of stationary multiaxial creep [11]

\[
\dot{\varepsilon}_{ij} = \frac{3}{2} \left( \frac{\sigma_e}{\sigma_c} \right)^{n-1} \frac{s_{ij}}{\sigma_c},
\]

(44)

where \( s_{ij} \) denote deviatoric stresses, \( \sigma_e = \sqrt{3/2}s_{ij}s_{ij} \) – the Huber–Mises–Hencky stress intensity, and \( \sigma_c \) – a material constant. Substituting Eq. (44) into Eq. (17) we obtain

\[
\Psi = \frac{3}{2} \left( \frac{\sigma_e}{\sigma_c} \right)^{n-1} s_{ij} \sigma_{ij} = \left( \frac{\sigma_e}{\sigma_c} \right)^{n+1} \sigma_c,
\]

(45)

since \( s_{ij} \sigma_{ij} = s_{ij} s_{ij} = \frac{3}{2} \sigma_e^2 \). This formula should be combined with Eq. (23) since tertiary creep was not taken into account. We obtain

\[
\dot{D} = \frac{1}{C} \left( \frac{\sigma_e}{(1 - D)^{\mu}} \right)
\]

(46)

with the constant \( \sigma_c \) included in \( C \). On the other hand, if we adjust Eq. (44) to allow for tertiary creep, introducing Eq. (5),

\[
\dot{\varepsilon}_{ij} = \frac{3}{2} \left( \frac{\sigma_e}{\sigma_c} \right)^{n-1} \frac{s_{ij}}{(1 - D)^{\beta} \sigma_c}
\]

(47)

and using Eq. (24) we arrive at

\[
\dot{D} = \frac{1}{C} \left( \frac{\sigma_e}{1 - D} \right)^{(n+1)/2}
\]

(48)

In view of Eq. (21) and usually accepted equality \( \mu = \nu \) we obtain in both cases

\[
\dot{D} = \frac{1}{C} \left( \frac{\sigma_e}{1 - D} \right)^{\nu}
\]

(49)

This is a generalisation of the Kachanov original uniaxial Eq. (2) to multiaxial stresses represented by the Huber–Mises–Hencky reduced stress \( \sigma_e \). Such an approach was experimentally confirmed for aluminium alloys Al–Mg–Si [10], but not for steel or copper. In these cases \( \sigma_e \) must be replaced by some other stress invariant, for example

\[
\sigma_{red} = a \sigma_I + \beta J_{1\sigma} + \gamma \sigma_c
\]

(50)

with \( a + \beta + \gamma = 1 \); in this equation \( \sigma_I \) denotes maximal principal stress (Galileo’s hypothesis), and \( J_{1\sigma} = \sigma_{kk} \) – the first stress invariant. Formula (50) was proposed by Hayhurst [10], but its particular cases earlier by Sdobyrev [35] with \( \beta = 0, a = \gamma = \frac{1}{2} \) and by Rabotnov [4] with \( \beta = 0, a = \gamma = 1 \).

So, for a broader class of materials Eqs. (46) or (48) do not hold. Whether the dissipated power approach may be sufficiently generalised in multiaxial cases, remains an open question.
6. Extension to biomechanics

Up to now, damage evolution equations were proposed for structural materials, it means for inanimate materials. Using thermodynamic approach it was shown that for such materials $\dot{D} \geq 0$ [5, 27], but Sidoroff added a remark “at least in an isothermal theory excluding recovery”. In the case of biological, living materials recovery (or related phenomena, called healing, repair, regeneration, restoration, maintenance) does take place and should be included in evolution equations. We also notice that the classical CDM, neglecting the above-mentioned effects, was successfully employed in biomechanics, mainly to investigate the implant-fixation endurance; several papers going this line were reviewed by Huiskes and Hollister [36].

Recovery resulting in decrease of damage (or microdamage, or microfracture, defined by Eq. (3) or other related formulae) was studied or just mentioned by many authors, mainly considering biomechanics of bones. Vernon-Roberts and Pirie [37] considered healing of trabecular microfractures in the bodies of lumbar vertebrae. Burr et al. [38] discussed effects of fatigue microdamage on bone modeling. Carter [39] noticed that bone remodeling is a continuous maintenance process that repairs the microcracks as they occur. Hanson et al. [40] considered creep damage of trabecular bone tissue and noticed that bone remodeling may repair this damage. Beaupre et al. [41] discussed restoration of a bone after cyclic loading: the predicted material density recovered but to a different distribution. Blackburn et al. [42] noticed that microcracks in trabecular bones can be repaired by microcallus formation similar to the healing response in long bones. Fatigue damage accumulation and repair in cortical bone was studied by Martin [43]. Prendergast and Taylor [44] proposed a form of bone maintenance theory: at remodeling equilibrium the rates of microdamage and its repair balance.

Considering skeletal muscles Zak [45] noticed that they regenerate fibres when injured. According to Vandenburgh et al. [46] temporary damage occurs to muscle cells under tension, and it may stimulate growth and remodeling.

In the present paper we propose a direct, simple generalisation of the evolution equations (6) and (24) to biological, living materials. Simple experimental data show that if the damage is relatively small, then the time of recovery is almost constant, hence the rate of recovery is proportional to the increment of damage from the initial state, $D_0$. For example, this observation holds for minor cut wounds. As $D_0$ we understand here the initial value of damage, before application of loads: in structural materials it may be due to oxidation (corrosion), Chaboche [12], in the case of growing tree – to rot, and in the case of a bone – to osteoporosis. However, if the damage is considerable, then we observe a remarkable increase of the time of recovery, meaning deviations from linearity. So, first we assume a generalisation of the second Eq. (6) to the case of biological materials in the form

$$\dot{D} = C_r \sigma^n - C_r [(D - D_0) - \frac{\varphi}{1 - D_0} (D - D_0)^2],$$

(51)

where $C_r$ denotes the modulus of recovery (with the dimension of reciprocal time), whereas $\varphi$ – dimensionless coefficient of nonlinearity of recovery. One may assume that $0 \leq \varphi \leq 1$; namely if $\varphi = 0$ then the recovery is linear, and if $\varphi = 1$ then it decreases to 0 for $D \rightarrow 1$, it means in the critical state. The value $\varphi > 1$ would mean vanishing regeneration even for $D < 1$ and this case is
rather without practical applications. The initial condition for Eq. (51) takes the form \( D = D_0 \) for \( t = 0 \), where \( t = 0 \) denotes the instant of load application.

Eq. (51) may be applied, for example, if we consider a damaged bone in vivo as a load-carrying structure. It is convenient to discuss the value of the coefficient of nonlinearity \( \phi \) taking total fracture of a bone (with dislocation) as an example. In this case, of course \( D = 1 \) at any point of the relevant cross section. If we assume \( \phi = 1 \) in Eq. (51) then even for \( \sigma = 0 \) we have \( \dot{D} = 0 \) and recovery (fusion of bone) cannot take place, hence \( \phi < 1 \). Just in the case of very advanced age of the patient we observe sometimes the inability of fusion – then the value \( \phi \) equal or close to unity is justified.

Eq. (51) contains five material constants, namely \( C_2, \mu, v, C_r \) and \( \phi \). This number is subject to essential reductions if we introduce specific dissipated power \( \Psi \), Eq. (17), and make use of Eq. (21). Suitable generalisation of Eq. (24) takes the form

\[
D = \frac{1}{C_d} \sqrt{\frac{\Psi}{1 - D} - C_r [(D - D_0) - \frac{\phi}{(1 - D_0)} (D - D_0)^2]].
\]

This equation contains just three materials constants, \( C_d, C_r \) and \( \phi \). Probably the constant \( C_r \) depends on the power \( \Psi \), supplied by the organism in the course of the process of recovery, but this dependence is not obvious and so we regard \( C_r \) just as a material constant.

Doubtlessly the evolution equations (51) and (52) proposed for biological materials, should be regarded just as an initial step in this direction. First, the variety of such materials may require individual approach in any case, or, at least, for particular classes of biological materials. Second, regeneration or recovery of damage in such materials may consist of several typical biological processes like growth (mass change), remodeling (property change) and morphogenesis (shape change) in adaptative form [47], and any of the processes may require a separate description. Moreover such phenomena as change of anisotropy during regeneration may be taken into account just introducing a tensorial damage parameter. Nevertheless, such evolution equations, much more complicated, will probably constitute suitable generalisations of the equations proposed. Of course experimental verification of damage evolution equations for biological materials in vivo (much more difficult than in the case of structural materials) will finally determine the value of individual proposals; unfortunately, most of the experiments were performed on devitalized materials and they do not allow to evaluate regeneration constants.

7. Conclusions

1. In the case of uniaxial stationary creep the Kachanov–Rabotnov evolution equation for scalar damage parameter \( D \) is expressed in terms of specific dissipated power \( \Psi \). For a relatively broad class of materials and temperatures the number of material constants may then be reduced.

2. The evolution equation derived is then postulated to cover more general cases of nonstationary creep, viscoplasticity and multiaxial states; in the last-mentioned case the application is restricted, for example to aluminium alloys.

3. Moreover, an extension to biological materials is proposed with recovery taken into account. In this case the rate of damage is not necessarily positive as it happens in structural, inanimate materials.
4. All extensions of the evolution equations for uniaxial tension, proposed in this paper, are subject to experimental verification. Such a verification in the case of biological materials tested in vivo may be particularly difficult.

5. This paper is devoted to creep damage only. An extension to cover also elastic and plastic effects of damage seems possible e.g. by using the general approach of Saanouni et al. [48].

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References


