On the concept of an equivalent column in the stability problem of compressed helical springs

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Summary: A more exact equivalent column for buckling of helical springs is introduced. It accounts for the pitch angle and possible buckling in two planes. Non-linear compression rigidity, local bending and shear rigidities as well as lower bounds for the mean values of these rigidities are established. The problem of anisotropy of buckling of helical springs is investigated in detail.

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1 Introductory remarks

A helical spring should be treated as a spatially curved bar. Such an approach, based on Kirchhoff-Clebsch equations, was developed by Nikolai [9] (cylindrical helix, non-linear treatment), Trostel [15] (arbitrary helix, linear treatment) and others. It was applied to the problems of stability of helical springs, e.g. by Chernyshev [3], Haringx [6], Olenov [10] (multiwire springs), Polishchuk [11], and Kernchen [8] (compression and torsion).

An exact stability analysis based on the theory of spatially curved bars is complicated and difficult for further applications. Hence, in most engineering applications a concept of an equivalent column is introduced. Such a column must account for compressibility of axis and shear effects.

First papers using the concept of an equivalent column to stability analysis of helical springs are due to Hurlbrink [7] and Grammel [4] and took into account only the effect of change in column length. They showed that there exists a limiting slenderness ratio below which no bifurcation is possible at all. Biezeno and Koch [2] pointed out that the effect of shear elasticity should also be taken into account. They overestimated it and in consequence they drew the conclusion that any spring, even short, would be liable to buckling. The correct solution under the approximate assumption of zero pitch angle (helix slope) was obtained by Haringx [5], Ponomariev [12] and others. It confirms the existence of a limiting slenderness ratio and is quoted in many monographs, e.g. Timoshenko, Gere [14], and Wahl [16].

Differences between Biezeno’s and Haringx-Ponomariev’s solutions result from different estimation of the shear effect. This problem was investigated in detail by Ziegler [17]. He showed that for helical springs a so-called modified approach is the proper way of taking shear force into account, whereas Biezeno and Koch [2] applied Engesser’s approach.

Equivalent columns were also introduced for buckling under combined compression and torsion (Ziegler, Huber [18], and Satoh, Kunoh, Mizuno [13]).

All the above-mentioned papers are based on the simplifying assumption of a very small pitch angle: this angle was assumed to be zero. A more general equivalent column allowing for
arbitrary helix slope is due to Berdychevsky and Sutyrin [1], but that concept proposed for general non-linear problems is rather complicated and has not been applied to stability analysis.

The aim of the present paper is to propose a more refined concept of an equivalent column which simplifies the investigation of buckling of arbitrarily shaped helical springs, but takes pitch angle and possible buckling in two planes into consideration and ensures high accuracy of solutions.

Because of the great importance of Haringx's paper [6] for our investigations we briefly discuss in the next section results obtained according to his equations.

2 Stability of helical springs according to Haringx

In the paper by Haringx [6] the helical spring is regarded as a wire. That highly compressible spring is made of ideal elastic material. Both ends of the spring are simply supported and free to rotate. The spring is axially loaded by a concentrated force.

Because of large displacements with respect to the helix in the unloaded state a non-linear precritical state is considered. The changes of radius, helix slope and number of coils are taken into account. That state is followed by bifurcation for which the additional displacements are assumed to be small and the linear theory of buckling is applied. Two functions of one variable describe the changes in shape of the compressed spring as an effect of buckling. According to the linear theory all higher order terms of these functions are ignored and finally one differential equation of fourth order is obtained. Making use of boundary conditions one algebraic equation which determines the buckling load is presented. Because of complicated form of that non-linear equation Haringx was able to give some series expansions only.

A detailed numerical investigation of the buckling load on the basis of Haringx's equation is done here.

A helical spring seems to be isotropic with respect to buckling under compression, but, in general, this is not true. It turns out that for constant helix slope not one but two curves re-
presenting the relation between critical compressive strain and slenderness ratio of the spring are obtained. That phenomenon is connected with the formation of preferred directions of buckling during deformation. A single (unique) solution (intersection of two curves) corresponds to points where the number of coils after compression \( n \approx 0.5, 1.0, 1.5, 2.0, \ldots \) slightly exceeds these values. The maximal relative differences of two compression strains are connected with \( n \approx 0.25, 0.75, 1.25, 1.75, \ldots \). This problem will be discussed in detail in Sec. 4.

The examples of those two curves for an initial helix slope \( \alpha_0 = 20^\circ \) are shown in Fig. 1 where \( \xi_{cr} \) denotes the critical compressive strain and \( R_0, H_0 \) the initial radius and height of the spring, respectively. Each curve has its own return point. Always the outer point with larger \( R_0/H_0 \) (limit value of \( R_0/H_0 \) ratio) corresponds to the limiting slenderness ratio \( H_0/R_0 \) below which no buckling occurs.

Such calculations were performed for various initial helix slopes \( \alpha_0, 0^\circ \leq \alpha_0 \leq 50^\circ \). The return points obtained for various \( \alpha_0 \) form the curves which are presented in Fig. 2. Those curves separate domains where springs never buckle (always stable regions marked by slanting lines) and the regions where such a phenomenon can occur. The boundaries of these regions are discontinuous functions of \( \xi_{cr} \), since the outer return point jumps from one curve to the other. Additionally, in Fig. 2 unstable regions are cut out by the condition \( \xi_{cr} < 1 \) (length of spring after compression cannot be smaller than zero). Theoretical solutions for \( \xi_{cr} > 1 \) are also shown in Fig. 2 by thin lines. More interesting — from the engineering point of view — is a presentation of the always stable regions in the space of geometrical parameters of a spring. It is done in Fig. 3. The regions where springs never buckle are marked by slanting lines as well; the thin lines correspond to the theoretical solution for \( \xi_{cr} > 1 \) (negative length of the spring at buckling). Similar to Fig. 2 discontinuities of \( R_0/H_0 \) can be seen here. For \( \alpha_0 = 0^\circ \) the limiting ratio is \( R_0/H_0 = 0.19082 \) and the corresponding critical compressive strain is \( \xi_{cr} = 0.8125 \) (for Poisson’s ratio \( v = 0.3 \)). However, on the contrary to common sense, those parameters describing the boundary of the always stable region can be larger, especially the compressive critical strain which can reach \( \xi_{cr} = 1.0 \). The highest limit value \( R_0/H_0 \) equals 0.1926 (the smallest slenderness ratio \( \lambda = 1/0.1926 \)) and corresponds to \( \alpha_0 \approx 12^\circ \).

Two additional lines are plotted in Fig. 3. The first of them, the higher one, which runs through the highest points of the graph can be called the pessimistic boundary of the always stable regions (upper bound of \( R_0/H_0 \) or lower bound of \( H_0/R_0 \)). The lower line is the optimistic boundary of
that domain because running through the lowest corners of the graph takes small parts of unstable regions as regions where springs never buckle. The real, complicated in shape, boundary of the stable region lies between those two curves.

3 The concept of an equivalent column

The application of an equivalent column simplifies the analysis of stability of springs. Such a column is assumed to be isotropic (equally possible buckling in any direction), and thus it is uniquely defined when its compression, shear and bending rigidities are given. The rigidities should be established in such a way as to ensure high accuracy of solutions. We demand that the relation between the critical strain $\xi_c$ and the ratio $R_0/H_0$ obtained using an equivalent column should form the envelope of exact solutions. Moreover, the return points, i.e. points belonging to the boundary of always stable region, should form an appropriate approximation of that domain. In the case under consideration we assume that our approximation is not lower than the so-called pessimistic boundary, Fig. 3. It means that solutions obtained using the equivalent column have some margin of safety.

Let us start from an equivalent column for a cylindrical spring, and next we will generalize the results to a spring of arbitrary shape.

We start with the determination of compression rigidity of a column. For a highly compressible spring the changes of geometrical parameters are considerable and the non-linear precritical state should be connected with the actual compression rigidity of the equivalent column. We assume that the actual rigidity is described by

$$EA = (EA)_0 \mu_e$$

where $(EA)_0$ is the initial rigidity in the unloaded state, and $\mu_e$ the modification function due to geometrical non-linearity.

First of all we determine the initial compression rigidity $(EA)_0$. Figure 4 shows the decomposition of generalized internal forces in a wire cross-section: bending moment $M_b$, twisting moment $M_t$, normal force $N$ and shear force $T$ are

$$M_b = PR_0 \sin \alpha_0, \quad N = P \sin \alpha_0,$$

$$M_t = PR_0 \cos \alpha_0, \quad T = P \cos \alpha_0.$$
Making use of Castigliano’s principle we calculate the deflection under force $P$ (regarded here as small, determined by linear elasticity)

$$f = \int_0^{2\pi n_0} PR_0 \left( \frac{R_0^2 \sin^2 \alpha_0}{EJ_w \cos \alpha_0} + \frac{R_0^2 \cos \alpha_0}{GJ_{0w}} + \frac{\sin^2 \alpha_0}{EA_w \cos \alpha_0} + \frac{\cos \alpha_0}{\gamma GA_w} \right) d\phi$$

(3.3)

where $EJ_w$, $GJ_{0w}$, $EA_w$, $\gamma GA_w$ are bending, twisting, tension and shearing rigidities of the wire, respectively, $n_0$ is the original number of coils, and $\gamma$ is the shearing factor for a circular cross-section. The last two terms in (3.3) due to tension and shear of the wire are very small and in further calculations they will be neglected. We assume that the equivalent column having the initial rigidity $(EA)_0$ and the height $H_0$ ($H_0$ is also length of the spring) has the same linearized deflection as the spring:

$$f = PH_0/(EA)_0.$$  

(3.4)

Comparing (3.3) and (3.4) we obtain

$$(EA)_0 = \frac{EJ_w H_0 \cos \alpha_0}{2\pi n_0 R_0^2 (1 + \nu \cos^2 \alpha_0)}$$

(3.5)

where $J_w = \pi d^4/64$ and $d$ is the diameter of the wire. For $\alpha_0 = 0$ (3.5) gives well known formulae quoted in [14], [16] and others. Using the simple geometrical relation

$$H_0 = 2\pi n_0 R_0 \tan \alpha_0$$  

(3.6)

we can rewrite (3.5) in the form

$$(EA)_0 = \frac{EJ_w \sin \alpha_0}{R_0^2 (1 + \nu \cos^2 \alpha_0)}.$$  

(3.7)

Now we pass to the non-linear analysis of compression in the precritical state. In view of the definition (3.1) of $\mu_0$ the actual compressive strain can be written as

$$\tilde{\varepsilon} \overset{\text{def}}{=} \frac{P}{(EA)_0 \mu_0}.$$  

(3.8)

On the other hand, assuming that the length of the wire is constant and utilizing geometrical relations we obtain

$$\tilde{\varepsilon} = 1 - \frac{\sin \alpha}{\sin \alpha_0}$$  

(3.9)
where $\alpha$ is the actual pitch angle after deformation. Eliminating $\xi$ from (3.8) and (3.9) we get

$$p = \left(1 - \frac{\sin \alpha}{\sin \alpha_0}\right) \mu_c$$

(3.10)

where $p = P/(EA)_0$. Further, following Haringx [6], the increment of curvature $k$ of the spring is proportional to the actual bending moment, i.e.

$$\Delta k = \frac{\cos^3 \alpha - \cos^3 \alpha_0}{R - R_0} = \frac{PR \sin \alpha}{EJ_w},$$

(3.11)

and the increment of the torsion $\tau$ is proportional to the twisting moment, i.e.

$$\Delta \tau = \frac{\sin \alpha \cos \alpha}{R} - \frac{\sin \alpha_0 \cos \alpha_0}{R_0} = -\frac{PR \cos \alpha}{GJ_{\omega w}},$$

(3.12)

where $R$ is the actual radius of the cylinder after deformation.

From (3.11) and (3.12) we obtain

$$p = \frac{1 + \nu \cos^2 \alpha_0 \frac{R_0^2 \cos^3 \alpha - R \cos^3 \alpha_0}{R_0^2 \sin \alpha_0}}{\sin \alpha_0 \frac{R^2}{\sin \alpha_0}},$$

(3.13)

and

$$\frac{R}{R_0} = \frac{(1 + \nu) \cos^2 \alpha + \sin^2 \alpha}{(1 + \nu) \cos^2 \alpha_0 + \sin \alpha_0 \cos \alpha_0 \tan \alpha}.$$  

(3.14)

Introducing (3.13) and (3.14) into (3.10) the modification function $\mu_c$ is determined: $\mu_c = \mu_c(\alpha, \alpha_0, \nu)$.

In order to describe buckling of a compressed helical spring we have to determine bending and shear rigidities (or compliances) of the equivalent column. They will be determined by a linear analysis, but in the deformed state, with $R_0$, $n_0$, and $\alpha_0$ replaced by $R$, $n$ and $\alpha$. Additionally, it is assumed that displacements caused by buckling are infinitely small and hence the changes of the geometrical parameters of a compressed spring can be neglected.

We start with the bending compliance determining first of all its local form. The compressed spring is loaded at its end by a concentrated overall bending moment $M^*$ which causes torsion of the spring wire, Fig. 5,

$$M_{tt} = M^* \cos \phi \cos \alpha$$

(3.15)
and double bending ($M_b = M_a + M_{tb}$)

$$M_b = M^*(\sin^2 \phi + \cos^2 \phi \sin \alpha)^{1/2}.$$  (3.16)

Using the energy approach (Castigliano's theorem) we calculate the increment of the total angle of deflection of the spring element

$$d\Theta = \frac{M^*}{EJ_w} (1 + \nu \cos^2 \alpha \cos \phi) ds$$  (3.17)

where $ds$ is the elementary length of the wire.

Next we define the local bending compliance $C_b$ demanding that the element $dx$ of the compressed equivalent column under moment $M^*$ has the same angle of deflection, namely

$$d\Theta = C_b M^* dx.$$  (3.18)

From that condition we get

$$C_b = \frac{1 + \nu \cos^2 \alpha \cos^2 \phi}{EJ_w \sin \alpha}$$  (3.19)

where $dx/ds = \sin \alpha$, and $C_b = C_b(\phi, \alpha, d, E, \nu)$ is called the local bending compliance because it refers to particular points of the column.

Now we are able to calculate the mean value of the bending compliance, namely

$$\bar{C}_b = \frac{1}{2\pi n} \int_{\phi_0}^{\phi_0 + 2\pi n} C_b \, d\phi = \bar{C}_b(\phi_0, n, d, \alpha, E, \nu) = 1/(EJ)_{\text{column}}$$  (3.20)

where $\phi_0$ represents the possible starting angular coordinate of the helix. Its effect will be discussed later.

The shearing compliance should be introduced also for the compressed spring with actual length $H$, pitch angle $\alpha$ and number of coils $n$. The compressed spring is loaded by a shear force $Q$. We are interested in displacement due to shearing, completely omitting deflection connected with bending of the spring as a whole. Therefore, in any cross-section, we take only the appropriate part $M_s$ of the total moment, namely

$$M_s = QR \sin \phi$$  (3.21)

which causes bending and torsion of the spring wire, Fig. 6,

$$M_b = M_s \cos \alpha, \quad M_t = M_s \sin \alpha.$$  (3.22)

Similar to the previous case we consider only an element of the spring with wire length $ds$ and referring to it an element of the compressed column $dx$. Making use of Castigliano's theorem we
can write
\[ d\phi = \frac{QR^2(1 + \nu \sin^2 \phi)}{EJ_w} \sin^2 \phi \, ds. \] (3.23)

On the other hand, the angle of shear deformation \( \chi \) of the column element under force \( Q \) and that found from Fig. 6 are equal:
\[ \chi = C_s \frac{d\phi}{dx}. \] (3.24)

Introducing (3.23) into (3.24) we have
\[ C_s = \frac{R^2}{EJ_w} \frac{(1 + \nu \sin^2 \phi)}{\sin \phi} \sin \phi, \] (3.25)

and \( C_s = C_s(\phi, \alpha, R, d, E, \nu) \) is the local shearing compliance of the compressed equivalent column. The mean value of \( C_s \) can be written as follows:
\[ \tilde{C}_s = \frac{1}{2\pi n} \int_{\phi^0}^{\phi^0 + 2\pi n} C_s \, d\phi = \tilde{C}_s(\phi_0, n, R, d, E, \nu) = \frac{1}{(GA)_{\text{column}}}. \] (3.26)

The mean values \( \tilde{C}_b \) and \( \tilde{C}_s \) depend on \( \phi_0 \) and \( n \) but if \( 2n \) is an integer they are independent of them. Then the critical loading found for such compliances is an average to both critical forces for the actual helical spring. Such a solution does not give any margin of safety. To obtain a lower estimation of the critical force we have to take into account that \( \tilde{C}_b \) and \( \tilde{C}_s \) are functions of \( n \) and \( \phi_0 \). In order to find that relation we rewrite (3.20) and (3.26) in the form
\[ EJ = \frac{1}{\tilde{C}_b} = \frac{2EJ_w}{2 + \nu \cos^2 \phi_0} \mu_b(n, \phi_0), \] (3.27)
\[ GA = \frac{1}{\tilde{C}_s} = \frac{2EJ_w}{R^2} \frac{\sin \alpha}{1 + \nu \sin^2 \alpha} \mu_s(n, \phi_0) \] (3.28)

where \( \mu_b(n, \phi_0) \) and \( \mu_s(n, \phi_0) \) stand for integrals in (3.20) and (3.26), respectively. In (3.27) and (3.28) we dropped the index column to simplify the notation.

It turned out that \( \phi_0 \) and \( n \) have very small influence on the bending rigidity \( EJ \) of the column. For \( \nu = 0 \) we get no influence at all \( (\mu_b = 1) \) and for \( \nu = 0 \) \( \mu_b \) is just slightly different from unity. Finally, for the sake of simplicity, we took with good accuracy \( \mu_b = 1 \).

The influence of \( \phi_0 \) and \( n \) on the shearing rigidity \( GA \) is much greater. According to (3.21) individual parts of the coil have various effects on the displacement. The regions closer to \( \phi = \pi/2 \) and \( 3\pi/2 \) (Fig. 6) give larger and regions closer to the force \( Q \) smaller contribution to the global shearing displacement whose increment is given by (3.23). The beginning of the spring does not have to start with angle \( \phi_0 = 0 \) and that possible starting angular coordinate has influence on the value of the integral in (3.26) and finally on \( \mu_s(n, \phi_0) \).

Assuming the most unfavorable situation (minimization of \( GA \) with respect to \( \phi_0 \)), it means the pessimistic solution, we obtained the estimation
\[ \mu_s = \max \left\{ 1 - 0.3576/\nu, 0.5 \right\}. \] (3.29)

The result presented above refers to cylindrically coiled springs. In the case of arbitrarily shaped springs the way of calculations is similar all geometrical parameters should be treated locally. After some calculations we obtained
\[ (EJ)_0 = \frac{EJ_w}{R_0^2} \frac{\sin \alpha_0}{\nu \cos^2 \alpha_0/(1 + R_0^2/R_0^2)} \] (3.30)
where (·)' denotes d(·)/dφ, and α₀, R₀, d may be functions of the independent variable φ. For the majority of springs the variations of geometrical parameters are small and the simplification 1 + R₀²/R₀ ≈ 1 is justified. Then the compression rigidity of an arbitrarily shaped spring is described by the same formulae as for a cylindrical spring but it should be treated locally.

For springs with considerable variation of R₀ (3.30) holds. The modification function μ₉ should be defined from conditions similar to (3.11) and (3.12) but they should be written for variable curvature and torsion of the spring.

The bending and shear rigidities for an arbitrary spring are described by (3.27) and (3.28), respectively.

4 Analysis of anisotropy of spring buckling

A helical spring, at least with the number of coils after precritical deformation n being an integer, seems to be isotropic with respect to buckling (equally possible buckling in any direction), but in fact it is not. This is clearly seen from the diagrams based on Haringx's equations, like in Fig. 1. We are now going to explain that phenomenon.

Obviously, local bending and shearing compliances Cₜ and Cₛ depend on φ, but their mean values Cₜ̄ and Cₛ̄ within the limits φ₀ ≤ φ ≤ φ₀ + 2πn do not depend on φ₀ if n is an integer or half of an integer. Nevertheless, the critical forces depend on φ₀ (on the direction of buckling), since bending moment and shearing force are not constant along the spring axis during buckling: for a simply supported spring the bending moment diagram is approximately sinusoidal, and that of the shearing force cosinusoidal.

The local bending compliance Cₜ changes very slightly with φ, and in the case v = 0 is even constant, whereas the local shearing compliance Cₛ is proportional to sin² φ, and hence it changes between zero and its maximal value. So, it is the effect of shear which causes the anisotropy of buckling for 2n being an integer. Figure 7 presents the parallel diagrams of Cₜ for φ₀ = 0 and φ₀ = π/2 and of shearing force Q in terms of φ (or of x). To have the picture as simple as possible we assumed the number of coils after precritical deformation n = 1, hence φ₀ ≤ φ ≤ φ₀ + 2π.

It is evident that in the first case (φ₀ = 0) the maximal shearing force meets the minimal local compliance (maximal local shearing rigidity), whereas in the second case (φ₀ = π/2) it meets the maximal compliance. Hence, the critical force corresponding to φ₀ = 0 is larger than that corresponding to φ₀ = π/2. Critical forces for both directions are equal to each other (isotropic case) for n slightly larger than an integer or half of an integer.

On the other hand, the equivalent column proposed in the present paper is isotropic with respect to the buckling direction: it gives a lower bound of the critical force for the actual helical spring.

\[ Cₜ \]
\[ Cₛ \]
\[ Q \]

Fig. 7. Diagrams of local shear compliance Cₛ and shear force
5 Application of the equivalent column to problems of spring stability

We are able now to calculate the buckling load for a spring of any shape. However, the comparison will be made for a cylindrical spring because for such a spring the exact results are known.

For a simply supported column the differential equations of buckling can be written in the form

\[ \frac{d\beta}{dx} = \frac{Pw}{EJ} \]

where \( dx \) denotes the actual length of a column element, \( w \) is the lateral displacement and \( \beta \) is inclination of the normal to the cross-section against the central axis of the undeformed column. In (5.1) the shear force was taken into account according to Ziegler's suggestion [17].

The appropriate boundary conditions

\[ w(0) = 0, \quad w(H) = 0, \quad (5.3) \]

lead to the solution

\[ \frac{P_c}{EJ} \left( 1 + \frac{P_c}{GA} \right) = \frac{\pi^2}{H^2} \]

which using (3.10), (3.13), (3.27) and (3.28) can be rewritten in the form

\[ f(\alpha_0) \frac{R_0}{H_0^2} = 1 + v \cos^2 \alpha_0 \]

where

\[ f(\alpha_0) = 2\pi^2 \frac{1 + v \cos^2 \alpha_0}{2 + v \cos^2 \alpha_0} \]

(5.6)

That relation with the geometrical constraint

\[ n = n_0 \frac{R_0 \cos \alpha}{R \cos \alpha_0} \]

(5.7)

and with (3.10), (3.13), (3.14) and (3.29) gives the actual critical helix slope \( \alpha_{cr} \) referring to the critical compression

\[ \xi_{cr} = 1 - \frac{\sin \alpha_{cr}}{\sin \alpha_0} \]

(5.8)

The results of numerical calculations for \( \alpha_0 = 20^\circ \) are shown in Fig. 1. The curve obtained for the equivalent column is an envelope of the exact solution and the accuracy in the whole interval is very high.

The boundary of the always stable domain for the column approach is presented in Fig. 3. That curve is higher than the so-called pessimistic solution but errors are small. The biggest discrepancy can be found for large values of \( \alpha_0 \), but such springs do not have practical application.

6 Conclusions

The proposed concept of an equivalent column is simple in practical application. Such an approach ensures high accuracy of solutions. It enables to solve problems of stability of arbitrarily shaped springs in a much simpler way. It also gives possibility to formulate optimization problems of springs with respect to their stability.
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