Generalized entropies: its connections with Shannon and Kolmogorov-Sinai entropies and an invariant based on this concept

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Let \((X, \Sigma, \mu)\) be a probability space
\(T : X \to X\) - (measurable) measure-preserving transformation
For a finite partition \(\mathcal{P} = \{E_1, \ldots, E_k\}\) we consider a join partition
\[
\mathcal{P}_n := \bigvee_{i=0}^{n-1} T^{-i} \mathcal{P} := \left\{ \bigcap_{i=0}^{n-1} A_i, \quad \text{where } A_i \in T^{-i} \mathcal{P} \text{ for } i = 0, \ldots, n-1 \right\}
\]
where
\[T^{-i} \mathcal{P} = \{T^{-i} E_1, \ldots, T^{-i} E_k\}.\]
Let
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\eta(x) = \begin{cases} 
0, & x = 0; \\
-x \ln x, & x \in (0, 1]. 
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Dynamical and Kolmogorov-Sinai entropy

Let

\[ H(\mathcal{P}_n) = \sum_{A \in \mathcal{P}_n} \eta(\mu(A)). \]

We define the entropy of the transformation \( T \) with respect to the partition \( \mathcal{P} \) (the dynamical entropy) as

\[ h_{\mu}(T, \mathcal{P}) = h(\mathcal{P}) = \limsup_{n \to \infty} \frac{1}{n} H(\mathcal{P}_n). \] (1)

For a given system \((X, \Sigma, \mu, T)\) we define the Kolmogorov-Sinai entropy of \( T \) (with respect to \( \mu \)) as

\[ h_{\mu}(g, T) = \sup_{\mathcal{P} \text{ finite}} h_{\mu}(T, \mathcal{P}). \] (2)
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Reasons of the generalization

- KS entropy is an isomorphism invariant; what about systems with equal entropy (e.g. zero entropy systems)
- how important are properties of $\eta(x) = -x \ln x$ for the entropy in dynamical systems
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Dynamical g-entropy

\[ G_0 = \{ g: [0, 1] \to \mathbb{R}, \ g - \text{concave}, \ g(0) = \lim_{x \to 0^+} g(x) = 0 \}. \]

Let \( g \in G_0 \). We define the g-entropy of the transformation \( T \) with respect to the partition \( \mathcal{P} \) (the dynamical g-entropy) as

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and $h(g_2, \mathcal{P}) < \infty$, then

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The behaviour of the quotient \( g(x)/\eta(x) \) as \( x \) converges to zero appears to be crucial for our considerations. Let

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\text{Ci}(g) := \lim \inf_{x \to 0^+} \frac{g(x)}{\eta(x)}, \quad \text{Cs}(g) := \lim \sup_{x \to 0^+} \frac{g(x)}{\eta(x)}.
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Define

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e.g. \( g(x) = \frac{x-x^\alpha}{\alpha-1}, \alpha > 1, \ g(x) = x \ln(1 - \ln x) \);

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\mathcal{G}_0^{\text{Sh}} = \{g \in \mathcal{G}_0 \mid 0 < \text{C}(g) < \infty\}, \quad \text{e.g.} \ g(x) = -x \ln \sin x;
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Corollary
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3. If \( g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{sh} \), then \( h(g, \mathcal{P}) = C(g) \cdot h(\mathcal{P}) \).
4. If \( g \in \mathcal{G}_0^\infty \) and \( h(\mathcal{P}) > 0 \), then \( h(g, \mathcal{P}) = \infty \).

Theorem
Let \( g \in \mathcal{G}_0^\infty \) and \( T \) be an aperiodic, surjective automorphism of a Lebesgue space \((X, \Sigma, \mu)\) and let \( \gamma \in \mathbb{R} \). Then there exists a partition \( \mathcal{P} = \{E, X \setminus E\} \), such that
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Measure-theoretic g-entropy

Following the Kolmogorov proposition we take the supremum over all partitions of dynamical g-entropy of a partition. For a given system \((X, \Sigma, \mu, T)\) we define

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h_\mu(g, T) = \sup_{\mathcal{P}\text{--finite}} h_\mu(g, T, \mathcal{P})
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Main theorem

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If $g \in G_0^0$, then $h_\mu(g, T) = 0$. If $g \in G_0$ is such that $Cs(g) = \infty$ and $T$ has positive measure-theoretic entropy, then $h_\mu(g, T) = \infty$.

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Let $g \in G_0^\infty$. If $(X, T)$ is aperiodic and surjective, then $h_\mu(g, T) = \infty$. 
Rates of g-entropy convergence

F. Blume “Possible rates of entropy convergence” Ergod. Th.& Dynam. Sys. 17. 45–70 (1997)

Let \((X, T)\) be a measure-preserving system, \(T\) –bijective, \((a_n)_{n \in \mathbb{N}}\) a monotone increasing sequence with \(\lim_{n \to \infty} a_n = \infty\) and \(c \in (0, \infty)\). Let \(P\) be a class of partitions of \(X\). Let \(g \in G_0\). We say that \((X, T)\) is of type \((LS(g) \geq c)\) for \((\langle a_n \rangle, P)\) if

\[
\limsup_{n \to \infty} \frac{H(g, P_n)}{a_n} \geq c \quad \text{for all } P \in P
\]

and \((X, T)\) is of type \((LI(g) \geq c)\) for \((\langle a_n \rangle, P)\) if

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for $g = \eta$ we obtain types of convergence introduced by Blume

this invariant was used for aperiodic, completely ergodic and rank-one systems (Blume 1997, 1998, 2000, 2011)

types $LS(\eta)$, $LI(\eta)$ were also used to distinguish some weakly mixing rank-one systems (Blume 1995)
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Choice of \( P \) and \( (a_n) \)

▶ class of partitions

\[
P(X) := \{ \mathcal{P} \mid \mathcal{P} = \{ E, X \setminus E \} \text{ for some } E \in \Sigma \text{ with } 0 < \mu(E) < 1 \}.
\]

If \((X, T)\) and \((Y, S)\) are isomorphic measure-preserving systems, then \((X, T)\) is of type \((\text{LS}(g) \geq c)\) for \(((a_n), P(X))\) iff \((Y, S)\) is of type \((\text{LS}(g) \geq c)\) for \(((a_n), P(Y))\).

▶ choice of \( (a_n) \)

if \((X, T)\) has zero entropy and \( g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{\text{Sh}} \), we have

\[
\lim_{n \to \infty} \frac{H(g, P_n)}{n} = 0
\]

for all finite partitions \( P \) of \( X \). Therefore we consider \( (a_n) \) such that \( \lim_{n \to \infty} \frac{a_n}{n} = 0 \).
Choice of $P$ and $(a_n)$

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for all finite partitions $P$ of $X$. Therefore we consider $(a_n)$ such that $\lim_{n \to \infty} \frac{a_n}{n} = 0$. 
Theorem
Let $g \in \mathcal{G}_0$ with $C(g) > 0$. If $(X, T)$ is an aperiodic measure-preserving system and $(a_n)$ is a positive monotone increasing sequence with $\lim_{n \to \infty} \frac{a_n}{n} = 0$, then $(X, T)$ is not of type $(\text{LI}(g) < \infty)$ for $((a_n), P(X))$. 
Can we get something new?

Table: Connections between $\eta$-entropy types and $g$-entropy types of convergence

<table>
<thead>
<tr>
<th>$\eta$-entropy</th>
<th>$g \in G^0_0$</th>
<th>$g \in G^{\text{Sh}}_0$</th>
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\( g_0 \in G_0^0 \) – aperiodic systems

- Every subshift over two symbols is of type \((\text{LS}(g) \leq 1)\) for \(((\varphi(2^{-n}))_{n=1}^{\infty}, P(X))\).
- Let \( g_0(x) = x \log_2(1 - \log_2 x) \).

**Theorem**

If \((X, T)\) is aperiodic and measure-preserving and \( \phi : [1, \infty) \mapsto (0, \infty) \) is an increasing function with \( \int_1^{\infty} \frac{\phi(x)}{x^2} \, dx < \infty \), then for every \( P \) such that \( \lim_{n \to \infty} \max\{\mu(A) | A \in P_n\} = 0 \), we have

\[
\limsup_{n \to \infty} \frac{H(g_0, P_n)}{\phi(ng_0(1/n))} = \infty.
\]

If \( \int_1^{\infty} \frac{\phi(x)}{x^2} \, dx = \infty \), then there exists a weakly mixing system \((X, T)\) and a meas. set \( E \) such that \( 0 < \mu(E) < 1 \) and \( \lim_{n \to \infty} \frac{H(g_0, P_n)}{\phi(ng_0(1/n))} = 0 \).
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\[ g_0 \in \mathcal{G}_0^0 \] – completely ergodic systems

- If \([0, 1], T) is completely ergodic, then there exists such a sequence \((a_n)\) with \(\lim_{n \to \infty} \frac{a_n}{n} = 0, \lim_{n \to \infty} a_n = \infty\), that for every \(\mathcal{P} \in \mathcal{P}([0, 1])\) we have
  \[
  \liminf_{n \to \infty} \frac{H(g_0, \mathcal{P}_n)}{a_n} \geq 1. 
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- Under the assumption of the previous theorem there exists \((a_n)\) such that \((X, T)\) is of type \(LS(\eta) = \infty\) for \(((a_n), \mathcal{P}([0, 1]))\).

- for every \(\mathcal{P} \in \mathcal{P}([0, 1])\)
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We may construct a class of rank-one weakly mixing systems where we can use type \((\text{LI}(g) \geq c)\) for \(((a_n), P(X))\) to distinguish systems. Depending on the choice of \(g\) we may use other than \(\eta\)-entropy types of convergence to differ rank-one systems.
Additional assumptions on $g$

$g'(0) = \infty \ (g \in G_0^0)$

Subderivativity of $g$

The crucial property of the static $g$-entropy is the following:

$$H(g, \mathcal{P} \vee \mathcal{Q}) \leq H(g, \mathcal{P}) + H(g, \mathcal{Q}|\mathcal{P})$$

It is sufficient that for every $x, y \in [0, 1]$ function $g$ fulfills the following condition

$$g(xy) \leq xg(y) + yg(x), \quad (5)$$

The condition is not easy to check. On the other hand if we want to construct such a function we can define

$$g(x) := xh(-\ln x),$$

where $h : (0, \infty) \mapsto \mathbb{R}$ is a concave, subadditive and increasing with $\lim_{x \to \infty} h(x) = \infty, \lim_{x \to \infty} \frac{h(x)}{x} = 0$. 
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Examples of subderivative functions

- for $h(x) = \ln(1 + x)$, we get $g(x) = x \ln(1 - \ln x)$,
- for $h(x) = x^\alpha$, $\alpha \in (0, 1)$ we have $g(x) = x(- \ln x)^\alpha$,
- if $h(x) := \begin{cases} x, & \text{for } x \in [0, 1) \\ 2^{-k}x + 2^{k+1} - 2, & \text{for } x \in [4^k, 4^{k+1}), \ k = 0, 1, \ldots \end{cases}$
then
$$g(x) = \begin{cases} 0, & \text{for } x = 0, \\ -2^{-k}x \log_2 x + x(2^{k+1} - 2), & \text{for } x \in \left(2^{-4^{k+1}}, 2^{-4^k}\right), \ k \\ -x \log_2 x, & \text{for } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$
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